# From the Abacus to the Digital Revolution
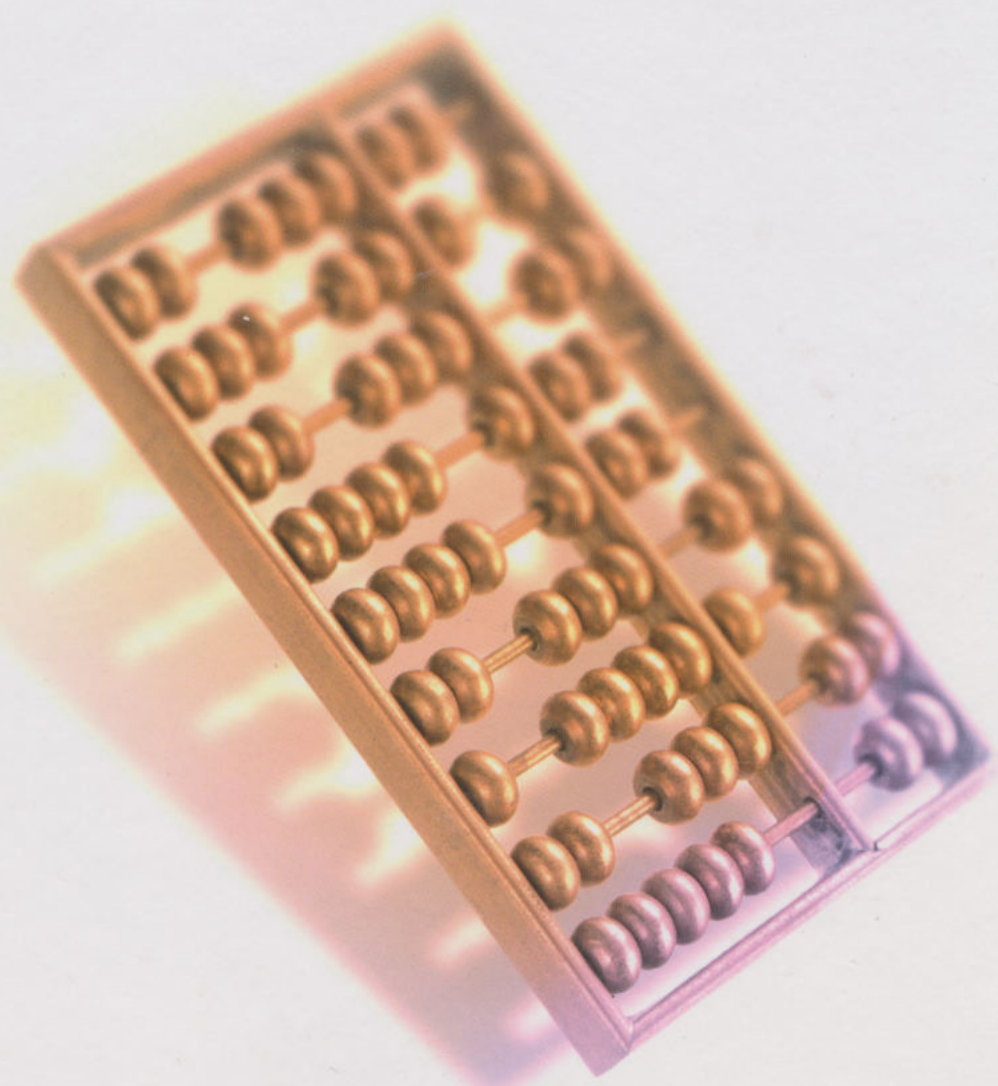
## Counting and computation

*Everything is mathematical*

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## Counting and computation

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*Everything is mathematical*

*Printed in Spain*

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# Preface

An algorithm is a method for automating a calculation. When given certain initial data it provides accurate results by following a series of rules in a predetermined order set out in a finite number of steps. Hence, an algorithm does not solve a single problem, but a series of problems of the same type, those governed by the same structure, regardless of the initial data. In the most commonly used sense, a formula is an algorithm. As such, the algorithm is a mathematical tool, but even this simple initial definition shows why it became central to computer science as well.

Algorithms are the small but powerful feats of mathematical engineering that operate inside the minds of the electronic equipment surrounding us and make our vibrant, digital world possible. In fact, a computer programme is nothing more than an algorithm written in a language that can be understood by a computer. However, the rise of algorithms through computing is only the most recent episode in a much longer and mathematically fascinating history that begins with the birth of calculation itself.

The relationship between calculation and technology is an age-old one. Throughout their evolution, calculating tools were always conditioned by the technology available at a given time, together with the number systems of individual cultures. The Egyptian methods for calculation and Roman calculating tools, such as the abacus, were largely determined by the numbering systems of each empire. In the case of Roman numerals, the use of the abacus for calculating helped them survive into the Middle Ages. The use of paper and pen for calculations produced a similar phenomenon, promoting the use of Arabic numerals. This process of evolution leads us in the end to computers. These were developed for the same purpose: to provide increasingly powerful tools for carrying out ever more complex calculations beyond the scope of pen and paper, abacus, or even the human memory.

The number $\pi$ offers a classic example of the evolution of calculation and its relationship to technology. Since the origins of mathematics in Mesopotamia and Egypt, attempts have been made to calculate the number $\pi$ using the instruments available to researchers at a given point. The results are amazing: Archimedes, in the 3rd century BC, had already provided an approximation with an incredibly low error of just 0.002. The development of computing has run in parallel to the search for more and more decimal places of $\pi$. At present, we have discovered millions of its decimal places and have techniques that allow us to calculate a digit at any given position in the number.

This book recounts the history of algorithms and computing and also describes the relevant aspects of calculation and its tools, from the prehistoric bones used for counting, through to the computers that dominate our world today. It is a fascinating and enlightening history, since it focuses on the material from which our present is made and asks how and why it has come to be as it is.

# Chapter 1

# The First Centuries of Computation: Positional Numbering

Calculation and computation are really synonyms, although due to the influence of English – the dominant language in the world of technology – the use of the word 'computation' has been reduced to its current meaning, related only to information technology. However, humankind has been inventing methods for computation, or rather calculation, for millennia.

The process began slowly, with the first step being the development of numerals. Like so many other cultural artefacts, numbering and calculation appear in different places, without connections, and then subsequently expand with great speed through a network of mutual influence. The history of numbers encompasses Mesopotamia, Egypt, Greece, Rome, India and other lands with their own types of computation, extending all the way to the appearance of positional numbering, or rather Arabic numerals, which represented a revolution in maths on the scale of Copernican changes to astronomy.

## The origins of numbers

The idea of a numbering system may be ancient, but it is neither universal nor uniform. Not all civilisations have developed numbers in the same way, and there are tribes – such as the Amazonian Pirahã – who do not have any numbering system at all. The oldest evidence of the use of numbers are bones with markings discovered in archaeological excavations. The oldest discovery to date is 35,000 years old. It is from the bone of a baboon discovered in the mountains of Lebombo in Swaziland in southern Africa during excavation work carried out in 1973. It has 29 marks and it is believed it was used to count the lunar phases, but it may also have been used to follow menstrual cycles. Its appearance is similar to the walking sticks that are still used to this day by the San, or Bushmen of the Kalahari region of Africa. Another

important discovery is a wolf's bone, found in 1937 in Vestonice in the Czech region of Moravia, which has 55 marks grouped into five groups of five and an additional mark after the 25 mark. It belongs to the Aurignacian culture and is 30,000 years old. An ivory Venus head was also unearthed in the surrounding area. The following important example is more recent. The Ishango Bone was discovered in the Congo in 1960 and is 20,000 years old.

The Ishango Bone, among the earliest archaeological evidence of the use of numbers, is displayed in the Royal Belgian Institute of Natural Sciences.

There are two theories regarding the origins of numbering, and these are also related to the question of which type of numbers appeared first: cardinal numbers (1, 2, 3…), or ordinal numbers (first, second, third,…). The most widely accepted theory argues along the lines of necessity. It all began with the need to count objects, and this led first to the invention of cardinal numbers, and then ordinal ones.

However, a second theory argues the spiritual origin of numbers, for use in rituals. Certain types of ceremonies would have required participants to move or position themselves in a pre-established order depending on the ritual, and as such ordinal numbers would have preceded cardinal ones. This latter theory postulates that numbers originated in a specific geographic location, from where they spread throughout the world. It also establishes the division of integers into even and odd, with odd numbers regarded as masculine and even numbers as feminine, a classification still common to many cultures throughout the world.

The use of ten digits, and the corresponding base 10, seems logical and natural to the modern Western mind, which may find it difficult to accept it is not universal. However, the evidence against it is irrefutable. Studies carried out with hundreds of different Native American tribes, for example, have shown that a wide range of bases

## THE PIRAHÃ

In the 1970s, the American missionary Dan Everett, currently one of the world's most eminent linguists, travelled to the Amazon to learn a rare language from the Pirahã tribe in conjunction with his evangelising. But after seven years of living with the tribe, Everett himself ended up losing his faith. The Pirahã are an extremely peculiar people. Their language, also known as Pirahã, does not feature subordinate phrases, in contrast to all known languages; furthermore, it has only ten phonemes. Stranger still, the Pirahã do not have myths or a collective memory, and refer only to events they have experienced themselves, or that have been experienced by someone known to them; nor do they think about the non-immediate future. However, perhaps most surprising of all is that Everett's studies, confirmed by psychologists, show that the Pirahã do not have any form of numbering system or calculation. For example, they do not distinguish between singular and plural, and make only a vague distinction between countable and uncountable objects. In fact, their calculation system is entirely approximate. Dan Everett recounts his experiences with the Pirahã tribe in the book *Don't Sleep, There Are Snakes: Life and Language in the Amazonian Jungle*, published in 2008.

for numbering are used (although certain bases were more common than others). Indeed, almost one third of cultures used the decimal system, but tribes who used a quinary (base 5) or even a quinary-decimal system make up a similar proportion. The remaining third is distributed between the dominant binary system (base 2; used by more than 20% of tribes) the vigesimal system (base 20; used by 10%) and a ternary system (base 30; used by 1%).

However, we do not need to draw on studies of exotic tribes to confirm our proposition. In the Indo-European language family, the word for 8 is actually derived from the word for 4. The Latin word *novem*, which corresponds to 9, appears to be derived from *novus* meaning 'new'. These examples suggests the use of base 4 and 8 numbering systems at some stage. Similarly, there are traces of a base 20 numbering system in the Basque words *hogei*, *berrogei*, *hirurogei* and *laurogei* (which mean 20, 40, 60 and 80, or more literally, 20, 2 · 20, 3 · 20, 4 · 20), and also in the French way of saying 80: *quatre-vingt*. Likewise, although corroborating the use of the decimal system, the English numbers contain traces of the ancient concepts: *eleven* and *twelve*, derived respectively from the expressions *one left*, and *two left*, (in the sense that they are leftover after the number 10).

## THE ORIGIN OF MATHEMATICS

The debate on the origin of mathematics is as old as the subject itself. Herodotus and Aristotle reflected at length on the subject. The former believed that geometry had appeared in Egypt in conjunction with re-establishing the partitioning of fields following the annual flooding of the Nile. As such, it was a practical tool. The latter, however, maintained that its appearance should be credited to priests, or more precisely, the legendary amount of free time they had. Hence, in his opinion, mathematics originated as an intellectual activity for the idle.



*The annual rise in the level of the Nile (shown passing through Luxor here) and the requirement to redraw the boundaries between flooded fields, formed the origin of mathematics, according to Herodotus, a Greek historian from the 5th century BC.*

Although it may seem that large numbers are very modern and only small numbers are represented in the texts and records left to us by history, this is not the case. The University of Oxford holds an Egyptian artefact that dates back some 5,000 years and records the victory of King Narmer over the Lebanese to the west of the Nile Delta. It describes Egypt as having taken 120,000 prisoners, 400,000 oxen, and 1,422,000 goats. Hundreds of thousands and millions are also mentioned in the Egyptian *Book of the Dead*.
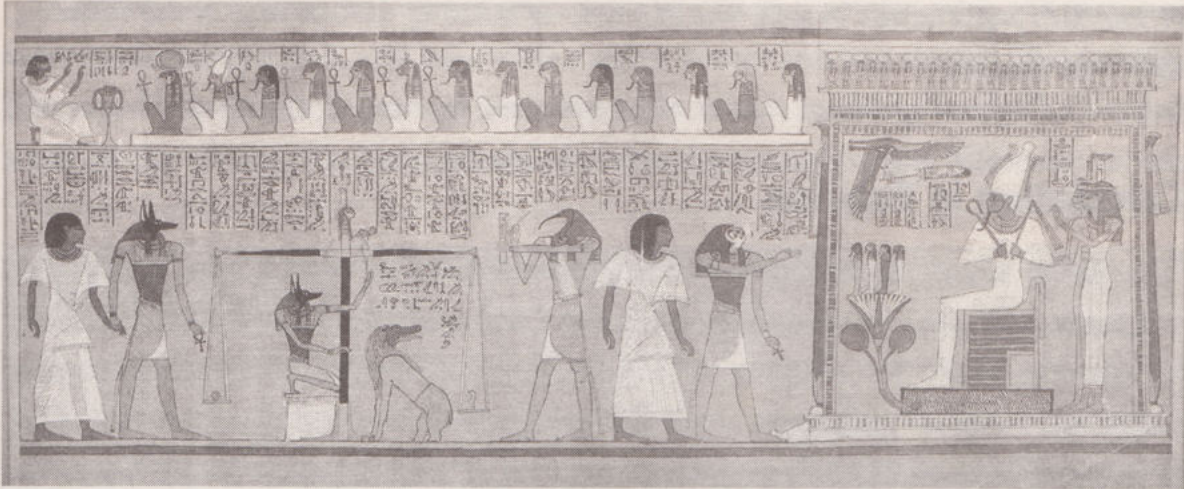
*Papyrus from the* Book of the Dead, *an Egyptian funerary text that includes references to large numbers.*

Even if the majority of cultures were aware of numbers (despite there being a wide range of bases), the same cannot be said for fractions, which were unknown to many. The Egyptians only considered fractions of the form $1/n$, and although the Babylonians had techniques that bear greater resemblance to those we know today, they used a sexagesimal notation system (base 60). The decimal notation currently used for fractions is a modern invention, promoted in large part by the work of the 16th century Flemish mathematician Simon Stevin.

Calculation techniques have developed throughout history in parallel to the technology available and the numbering system in use at a given time. Papyrus had a direct influence on the numbering system and calculation of the Egyptians, which was developed in such a way that it could be used easily on that medium. The Roman numbering system, in contrast, was difficult

*Page from Simon Stevin's work* De Thiende, *published in 1585, in which the Flemish mathematician proposed a new notation for writing decimal numbers.*

to use with papyrus, and as such was based on calculating devices such as the counting stick and abacus.

## Calculation in Babylon

The region adjoining the Tigris and Euphrates valleys, in what is now largely Iraq, was known by the Greeks as Mesopotamia, a word which literally means 'the land between the rivers'. The confluence of these two great rivers made the area extremely fertile, and some of humanity's oldest and most refined civilisations flourished there. Writing appears in the area from around 3000 BC and is linked to the Sumerian culture. Its first manifestations took the form of pictograms – graphical representations of the object being denoted – which then evolved into cuneiform writing. Once again, the change was a result of the influence of technology; the new system of writing was defined by the materials that were used.

*Example of cuneiform writing on a Sumerian clay tablet, dated to 2600 BC. The tablet is a contract for the sale of a house and the neighbouring land.*

Cuneiform writing was carved onto wet clay tablets. To begin with, the incisions were made using a reed, and later with a stylus whose end was shaped like a wedge (the word 'cuneiform' is derived from the Latin *cuneus*, meaning 'wedge'). Detailed study of the writing has been possible because, fortunately, many Sumerian tablets

## THE CITY OF URUK

Uruk was an ancient Mesopotamian city located on the banks of the Euphrates, some 225 kilometres from what is now Baghdad. At the height of its glory, towards the third millennium BC, it was the largest city in the world. The Sumerian tradition describes it as the birthplace of the hero Gilgamesh, the protagonist in one of history's oldest sagas. Uruk is also considered the cradle of calculation and counting. Some scholars maintain that the modern name Iraq is derived from the Sumerian name Uruk, although the theory remains a controversial one.



*The archaeological centre of Uruk, regarded as the birthplace of calculation and counting.*

have survived to the present day in good condition. In fact, there are currently 400,000 clay tablets from Mesopotamia held in museums throughout the world. Some 400 of these have content related to mathematics, the oldest coming from the lost city of Uruk.

The numbering and calculation system that developed in Mesopotamia lasted for a long time, transcending the confines of Sumerian culture. Although outside cultures imposed themselves politically on the region, attracted by the prospect of its wealth and power, it was these invaders who were fundamentally changed, converted by the advanced Mesopotamian culture, including its mathematics.

*Cuneiform representation of Babylonian numbers.*

The Babylonians used a sexagesimal, or base 60, numbering system, which means that each digit corresponds to a number between 0 and 59. It was also a positional system, which means that the same digit can represents a different number depending on its position; in the case of the sexagesimal system, a power of 60. To illustrate this concept, let's consider the number represented by the three sexagesimal digits 3, 3, 3 as an example. In this number, the value of the number three depends on its position. The first 3 corresponds to the value $3 \cdot 60 \cdot 60$, the second to $3 \cdot 60$ and

the third to 3. Hence, expressed in decimal (base 10) numbers, the number would be $3 \cdot 60 \cdot 60 + 3 \cdot 60 + 3 = 10{,}983$. Let us consider a more complicated example: the number 23, 4, 52, which represents $23 \cdot 60 \cdot 60 + 4 \cdot 60 + 52 = 83{,}092$. This example shows us that, when we use a given base, the digits must be less than the number of the base. For example, in the sexagesimal system, the digits are less than 60. Finally, it is also worth noting that, although this numbering system is base 60, the basic groupings use the decimal system, or rather are grouped in multiples of 10.

| 1 | geš (or aš or diš) | 10 | u | 60 | geš |
|---|---|---|---|---|---|
| 2 | min | 20 | niš | 120 | geš-min |
| 3 | eš | 30 | ušu | 180 | geš-eš |
| 4 | limmu | 40 | nišmin (or nimin or nin) | 240 | geš-limmu |
| 5 | iá | 50 | ninnu | 300 | geš-iá |
| 6 | àš | 60 | geš (or gešta) | 360 | geš-àš |
| 7 | imin | | | 420 | geš-imin |
| 8 | ussu | | | 480 | geš-ussu |
| 9 | ilimmu | | | 540 | geš-ilimmu |
| | | | | 600 | geš-u |

*The names of Babylonian numbers.*

There are a number of theories that attempt to explain the origin of this system. In 1904, Kewitsch, a scholar who studied the Assyrian civilisation, published his proof that the sexagesimal system was a mixture of two previous systems, one that used base 6 and another that used base 10. However, Georges Ifrah, whose *The Universal History of Numbers* was published much more recently and is regarded as the authority on the subject, believes the base 6 theory lacks evidence, since such a system has hardly ever been used. His theory is that base 60 originated as a combination of base 12 and base 5. The use of both bases is well documented and, in fact, if we study the Sumerian numbers from 1 to 10, we are given the impression that the numbers 6, 7 and 9 contain traces of the base 5.



*The cover of the ground-breaking book* The Universal History of Numbers, *written by the French mathematics historian Georges Ifrah in 1985.*
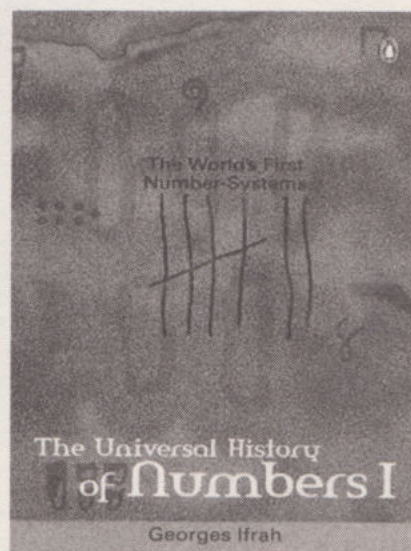
The following table reproduces Ifrah's work and shows that *iá*, the Sumerian word for 'five,' is used as the basis for representing other numbers by means of addition. Hence, the Sumerian number 6 is *iá.geš*, where *iá* is 5 and *geš* is 1. Hence in Sumerian the number 6 is expressed using the composite word 'five–one,' which appears to suggest an addition.

| 1 | geš (or aš or diš) | 6 | àš < iá.geš |
| 2 | min | 7 | imin < iá.min |
| 3 | eš | 8 | ussu |
| 4 | limmu | 9 | ilimmu < iá.limmu |
| 5 | iá | 10 | u |

The positioning of the Babylonian system was extremely flexible. A triplet (a, b, c) could represent both $a \cdot 60^2 + b \cdot 60 + c$ as well as another relationship with another three consecutive powers of 60, for example, $a + b \cdot 60^{-1} + c \cdot 60^{-2}$. Fractions can be represented using negative powers, as in the second example, and so the number

## A MATTER OF THE ANCIENTS

Perhaps readers have already realised that the sexagesimal system is not dead and buried in the sands of the ancient Middle East, but is present in our daily life. We make use of it not just every day but every minute. We have a base 60 system when it comes to dividing time. It was the Babylonians who divided the day into 24 hours, the hour into 60 minutes and the minute into 60 seconds. The present system used for describing angles follows the same pattern and was also established by the Babylonians,

*The astronomical clock in the Italian city of Brescia.*

$(1, 2, 3)$ corresponds to $1 + 2/60 + 3/(60 \cdot 60)$. As such, this notation allows fractions to be represented to a high degree of precision. However, in spite of this quality, positional representation presents an awkward problem. The absence of a value must also be represented, a tricky problem for a system that did not yet include zero. The Babylonians used a specific symbol to represent a number with no value. (Around 130 AD, in Alexandria, Egypt, Ptolemy would use omicron to represent zero, the fifteenth letter of the Greek alphabet that is similar in appearance to our letter o.)

Babylonian numbering did not fully sever its ties with the pictographic writing that preceded it and which are visible in the design of its representations. The Babylonians had their own calculators, a somewhat basic system that was nonetheless highly effective. It was based on the manipulation of different objects that represented different quantities: One object represented the unit, another a ten, another sixty, etc. These objects made it easy to carry out complex calculations as part of the everyday transactions of Babylonian life. Indeed, it appears the first written notations took their shapes from these original physical objects.

Various discoveries have corroborated this theory. From the outset in 1896, the excavations of the Palace of Nuzi, 90 kilometres from the Tigris, very close to the current northern Iraqi city of Kirkuk, saw the discovery of 5,000 cuneiform tablets from the 15th and 14th centuries BC, and around 200 from the 24th and 23rd centuries BC. A clay vessel in the form of an egg-shaped pouch, which contained a series of identical spherical figures, was also found among the remains. On the



*The famous Plimpton 322 tablet, dated to between 1824 and 1784 BC, shows a series of numbers in four columns, written using sexagesimal notation.*

outside, the vessel had an inscription that listed a certain number of units of livestock – exactly the same number of clay figures that were contained inside. In addition to this, in Susa, a city located in what is now Iran and regarded as one of the oldest settlements in the world, similar clay pouches were found containing various discs, cones, bowls and sticks that could be matched with the written numerals once the values had been deciphered.

The Babylonian system for calculating was highly advanced, as can be seen from the large number of tablets that contain mathematical information. Many of these contain tables. Tables have been discovered that were used for multiplication, squaring and cubing a number. Reciprocal tables have also been discovered. In a number of the latter tables, the reciprocals for 7 and 11 have been missing, among others. In base 60 those numbers would be represented with an infinite number of digits. On other tables approximations that are higher and lower than these exact reciprocals are shown. A number of tablets contain the square roots of numbers and their powers. It is believed that the tables of powers were used to calculate the inverse of exponentials (logarithms), and that, when the inverse of a given number was unavailable, it was estimated by means of the linear interpolation of the numbers that appeared in the table.

A multiplication table for the number nine is reproduced below from a tablet discovered in Nippur and held at the University of Jena. The adaptation to the current numbering system has been carried out by the historian of mathematics and ancient sciences, Christine Proust. The table presents some interesting features. For example, 1, 3, which correspond to the multiplication $9 \cdot 7$, is understood as $1 \cdot 60 + 3 = 63$; and 7, 30, which corresponds to $9 \cdot 50$, is understood as $7 \cdot 60 + 30 = 420 + 30 = 450$.

| | | |
|---|---|---|
| $9 \cdot 1 = 9$ | $9 \cdot 11 = 1, 39$ | $9 \cdot 30 = 4, 30$ |
| $9 \cdot 2 = 18$ | $9 \cdot 12 = 1, 48$ | $9 \cdot 40 = 6$ |
| $9 \cdot 3 = 27$ | $9 \cdot 13 = 1, 57$ | $9 \cdot 50 = 7, 30$ |
| $9 \cdot 4 = 36$ | $9 \cdot 14 = 2, 6$ | |
| $9 \cdot 5 = 45$ | $9 \cdot 15 = 2, 15$ | |
| $9 \cdot 6 = 54$ | $9 \cdot 16 = 2, 24$ | |
| $9 \cdot 7 = 1, 3$ | $9 \cdot 17 = 2, 33$ | |
| $9 \cdot 8 = 1, 12$ | $9 \cdot 18 = 2, 42$ | |
| $9 \cdot 9 = 1, 21$ | $9 \cdot 19 = 2, 51$ | |
| $9 \cdot 10 = 1, 30$ | $9 \cdot 20 = 3$ | |

The following example, which was also adapted by Proust, is a table of inverses that was discovered in Nippur. In this table, the number 20 refers to $20 \cdot 60^{-1} = 20 / 60 = 1/3$.

| | | | |
|---|---|---|---|
| $1/2 = 30$ | $1/10 = 6$ | $1/20 = 3$ | $1/40 = 1, 30$ |
| $1/3 = 20$ | $1/12 = 5$ | $1/24 = 2, 30$ | $1/45 = 1, 20$ |
| $1/4 = 15$ | $1/15 = 4$ | $1/25 = 2, 24$ | $1/48 = 1, 15$ |
| $1/5 = 12$ | $1/16 = 3, 45$ | $1/27 = 2, 13, 20$ | $1/50 = 1, 12$ |
| $1/6 = 10$ | $1/18 = 3, 20$ | $1/30 = 2$ | $1/54 = 1, 6, 40$ |
| $1/8 = 7, 30$ | | $1/32 = 1, 52, 30$ | |
| $1/9 = 6, 40$ | | $1/36 = 1, 40$ | |

To calculate square roots, the Babylonians applied an algorithmic method now known as the bisection method. The method has been attributed to many philosophers and mathematicians, such as the Greek Archytas and Heron of Alexandria. It is also known as 'Newton's method,' although we know it was originally Babylonian.

Given a number $N$, of which we wish to calculate the square root, two approximations are made, $a_1$ and $b_1$, which are greater than and less than $N$, respectively. We then calculate $a_2 = (a_1 + b_1)/2$ and check if the square is greater than or less than $N$. If the value is higher, $a_2$ will replace the original higher value; if it is lower, $a_2$ will replace the other value. The process is repeated until finding a value whose square is $N$, or which provides a good enough approximation.

The Babylonians could also solve systems of equations and second degree equations for problems with no complex solutions. Such problems appear in texts from the same era, around the year 2000 BC. These Babylonian "problems" also solved certain types of third degree equations. Equations of the form $x^3 = a$, and $x^3 + x^2 = c$ are solved using tables, and those with the more complicated form $ax^3 + bx^2 = c$ are reduced to the first types.

From analysing the Babylonian texts, it is clear that for them, mathematics was more than just a practical tool, something that represented a fundamental difference from the Egyptians, who had a much more utilitarian concept of the subject. On the contrary, the Babylonians were highly advanced in the development of arithmetic and algebra. However this was not the case with geometry – literally "measuring the earth" – in which the Egyptians led the way. In terms of geometry, Babylonian knowledge was rudimentary and limited to the characteristics of a few simple shapes such as triangles and quadrilaterals.

## SECOND AND THIRD DEGREE EQUATIONS

Second degree equations, of the form $ax^2 + bx + c = 0$, are commonly resolved using the following formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which provides one real solution for positive or zero discriminants ($x$), or rather, when $b^2 - 4ac$ is greater than or equal to zero.

To solve $ax^3 + bx^2 = c$, the Babylonians multiplied the equation by ($a^2/b^3$), to give $(ax/b)^3 + (ax/b)^2 = ca^2/b^3$. The resulting equation could be solved using tables of the form $x^3 + x^2 = c$, then calculating the value of $x$.

However, Babylonian work on the circumference of a circle has lasted to the present day. It was the Babylonians who divided it into 6 parts based on the radius, and each part into 60, thus obtaining 360°. Degrees are then subdivided into 60 minutes, which in turn are divided into 60 seconds. The number $\pi$ was approximated as 3, although a tablet discovered in Susa compares the ratio between the perimeter of a hexagon and the circumference of a circle, giving a better value of 3 1/8 for $\pi$.
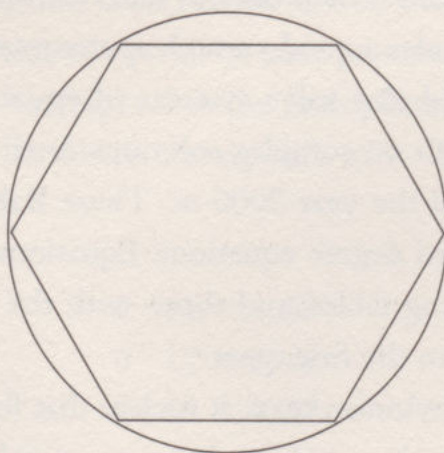
*Construction of a hexagon inscribed within the circumference of a circle, based on the radius of the circle.*

## Calculation in Egypt

The numbering system of ancient Egypt made use of a symbol for each power of ten. Hence, there was a system for units, another for tens, another for hundreds, etc.

In contrast to the Babylonian system, the Egyptian system was not positional. The hieroglyphs that represented the most common numbers are shown below:

| Value | Hieroglyph |
|---|---|
| 1 | |
| 10 | |
| 100 | |
| 1,000 | |
| 10,000 | |
| 100,000 | |

The Egyptian numbering system used additive notation, in contrast to ours, which, like the Babylonian system, is positional. The number 3,204, written using additive notation, for example, would be 1,000 + 1,000 + 1,000 + 100 + 100 + 1 + 1 + 1 + 1, which, using Egyptian hieroglyphs, would be written:
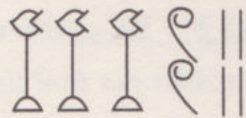
This system made it possible to represent large numbers, as well as facilitating addition and subtraction. For addition, where necessary, surpluses were 'carried' over to larger units, and for subtraction, numbers were 'taken' from these. Multiplication was reduced to addition and subtraction, and was carried out using an ingenious but nonetheless complicated method.

Let's illustrate the process with an example, the multiplication of 17 and 53. Take the pair 1 and 53 and double each number to give 2 and 106. Repeat the operation, to give the pair 4 and 212. Keep doubling the two values until the first number exceeds 17, at which point stop the process, discarding the last result. Hence, we now have the following pairs:

| | |
|---|---|
| 1 | 53 |
| 2 | 106 |
| 4 | 212 |
| 8 | 424 |
| 16 | 848 |

It is now necessary to discover how the number 17 can be obtained by adding the values in the first column. In this instance, the only way of obtaining 17 is by adding 1 and 16. Hence, the values that correspond to 1 and 16 should be added together, i.e. 53 and 848. The sum is 901. Hence, the result of multiplying 17 by 53 is 901.

| | | |
|---|---|---|
| 1 | 53 | ◄ |
| 2 | 106 | |
| 4 | 212 | |
| 8 | 424 | |
| 16 | 848 | |
| 17 | 901 | = Result |

This way of operating corresponds to decomposing the number 17 into base 2, before taking the products that correspond to 53. Hence, the decomposition of 17 is $17 = 2^0 + 2^4$, and the addition selects the values $(2^0 + 2^4) \cdot 53$, disregarding the other products, $2^1 \cdot 53$, $2^2 \cdot 53$ and $2^3 \cdot 53$, as these do not form part of the binary decomposition of 17. The procedure works in a similar way to calculations carried out by computers. It works because the decomposition of a number into powers of 2 is unique, there is only one selection of values from the example that add up to 17. Hence, the set of values in the right-hand column that are added together is also unique. This method of multiplication is perhaps better known as Russian multiplication.

Division is calculated as the inverse operation of multiplication. To illustrate that process, let's use the same numbers and divide 901 by 17, which will give 53. This is a division that gives a whole number, without a remainder or decimal places.

Taking the denominator, 17, and the value 1, like in the above process, both values are doubled to give 34 and 2. The process is repeated, to give 68 and 4, and so on while the first value is less than the numerator (in this case 901).

When the first value exceeds the numerator, that final pair is discarded, giving the following list of pairs:

| 901/17 | 17 | 1 |
|---|---|---|
| | 34 | 2 |
| | 68 | 4 |
| | 136 | 8 |
| | 272 | 16 |
| | 544 | 32 |

We have discarded the next pair (1,088 and 64) since it is greater than 901. Now we must determine the values in the first column that add up to 901. In this example, the values are 544, 272, 68 and 17 (since 544 + 272 + 68 + 17 = 901). The sum of the corresponding values from the right column will give the final result. Hence, 32 + 16 + 4 + 1 = 53.

| 901/17 | 17 | 1 | ◀ |
|---|---|---|---|
| | 34 | 2 | |
| | 68 | 4 | ◀ |
| | 136 | 8 | |
| | 272 | 16 | ◀ |
| | 544 | 32 | ◀ |
| | ——— | | |
| | 901 | 53 | = Result |

As was the case for multiplication, the decomposition of 901 is unique. It has been calculated in terms of the number 17 multiplied by the powers of 2 that add up to 53. In this case, the result of the division is a whole number, but when this was not possible and the result gave a remainder, the Egyptian procedure made use of fractions. However, the use of Egyptian fractions was more complicated than the current system, because with few exceptions the Egyptians only used fractions of the form $1/n$, i.e. fractions with a numerator of 1. What is curious, however, is that this was due to a limitation caused by the way the fractions were expressed: A symbol was used to denote the fraction, followed by the symbols corresponding to the value of the denominator. No information was provided regarding the numerator, which could only ever be 1.

The Egyptians used the following symbol to indicate a fraction:

And then added the denominator, in this example 21:

Hence, the notation above represents the fraction 1/21.

It has been noted that there were fractions with a numerator that was not 1. This was the case with the fraction 2/3, which had its own symbol, and the fraction $n/(n + 1)$ that corresponds to the inverse of $1 + 1/n$. Or rather, $1/(1 + 1/n) = 1/((n + 1)/n) = n/(n + 1)$.

The importance of fractions and their treatment can be seen clearly in the Rhind Papyrus (also known as the Ahmes Papyrus) which begins by describing the decompositions of the fraction $2/n$ using expressions of the form $1/x + 1/y + ... + + 1/z$ for all the odd numbers between 5 and 101, and then describes expressions for the fractions $n/10$ for $n = 2$ up to 9.

## THE RHIND PAPYRUS

This 6-metre long Egyptian papyrus is famous for its mathematical content; 87 widely ranging problems together with their solutions. It is dated to between 2000 and 1800 BC, but its author, Ahmes, explains that it represents knowledge that was over 200 years old, gathering it together in a book for training future scribes. As such, it can be considered a primitive "textbook" for teaching mathematics. It has been held in the British Museum, since 1858, but was originally part of the collection of Scottish Egyptologist Henry Rhind, hence the name.
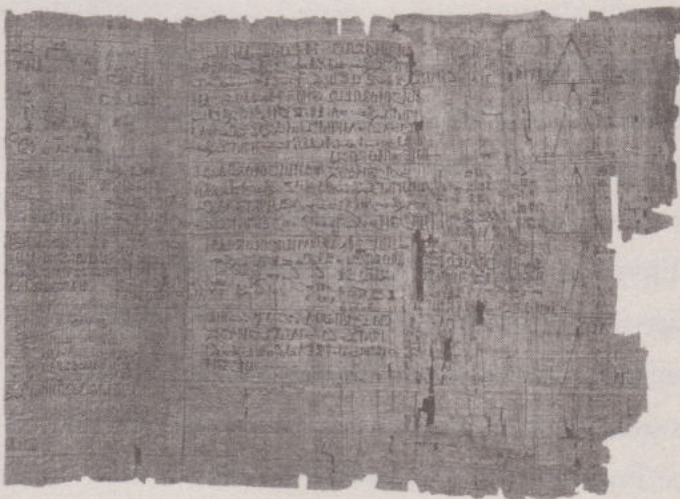
In addition to setting out the workings of Egyptian fractions for posterity, the Rhind Papyrus also gives an idea of the type of everyday problems that had to be solved at that time and how they were carried out. The first problems in the document are divisions by 10, for which the previously mentioned table for $n/10$ was used. These are followed by arithmetic and geometry problems, but also by problems that would be understood as linear equations today, now written in the form $ax + bx = c$. Some of the problems from the papyrus include squares of unknowns (when expressed using modern terms), however in spite of this, it is thought the Egyptians did not know how to solve second or third degree equations.

The majority of the problems on the papyrus are solved using a procedure now known as the false position method, or *regula falsi*. Only the papyrus's problem 30 is solved using the same technique as employed today, namely by factorising and dividing. To understand the false position method, let's look at problem 24 as an example, which we would now solve using a linear equation. The problem is as follows:

'Determine the price of one pile if one pile and a seventh of a pile is 19.'

Expressed in current notation, the problem is, $x + 1/7 \, x = 19$.

The false position method entails assuming a value for the unknown, and calculating the expected result for this value. Since the assumed value will be incorrect, the resulting value will also be incorrect, but will then be subsequently modified to give the correct result. Hence, a price is assumed for the pile. In this case let's see if $x = 7$. The resulting value for one and a seventh of a pile will be 8. Hence $x + 1/7 \, x = 8$ for $x = 7$. From here, we determine how to modify the selected value, 7, to give 19 instead of 8. The number 8, or another value of $x$, is multiplied by 19/8. Using only fractions whose numerator is 1, we get $2 + 1/4 + 1/8$ is 19/8. Multiplying 7

## THE NUMBER π IN EGYPT

The Rhind Papyrus provides the oldest approximation of the number $\pi$, slightly above the value we know today. The value is given as 256/81, or 3.1604. It may be the oldest approximation, but it is not the most precise. Indeed, subsequent Egyptian documents contain more precise approximations; the closest is $3 + 1/7$.

by (2 + 1/4 + 1/8) gives 16 + 1/2 + 1/8. The papyrus also shows that the solution is correct, calculating that this value plus one seventh of the value equals 19.

All these calculations could be carried out thanks to the introduction of papyrus, which made it possible to overcome the limitations of media such as wet clay, wax, etc., on which it would have been extremely complicated and uncomfortable to carry out these operations. In this way, the Egyptians were able to work in a similar way to using paper. Hieratic writing was developed for writing on papyrus, a practical simplification of hieroglyphic writing for use by scribes in administrative roles. Later on, demotic writing appeared, an abbreviation of hieratic writing, which, as indicated by its name, was the common form of writing used for dealing with everyday matters, with the hieratic system left for sacred matters. This process of simplification was also responsible for changes in numerical notation that would pave the way for numerical digits.



*The Ebers Papyrus (left), dated to the 16th century BC and dealing with medicine, is an example of hieratic writing, whereas the Rosetta Stone, from the 2nd century BC contains three types of writing: hieroglyphic, demotic and Greek.*

Demotic writing overcomes the original problem of Egyptian notation by including a numerical symbol for each power of 10. This implied that the representation of 9 corresponded to 9 times the mark used for the unit and, similarly, the representation of 99 to 9 times the symbol for ten and 9 times that of the unit. Demotic writing created a mark for each value from 1 to 9, a mark for each ten from 10 to 90, and likewise for the other powers of 10. To represent the values, it

was necessary to memorise all the corresponding symbols. However, in spite of apparently lacking rigour, the memorising of drawings was not strange for Egyptian mathematicians. In Egypt, there was no unified notation for mathematical operations; the Rhind Papyrus made use of perpendicular lines in different positions to represent addition and subtraction.

## Greece

Greek mathematics was built on Babylonian and Egyptian foundations. Egyptian mathematical techniques reached Greece through the commercial exchanges between both lands, which peaked between 700 and 600 BC. That period represented the golden age of the exchange of knowledge, an age in which many Greek mathematicians travelled to Egypt and learnt the secrets of millennia-old knowledge.

It was perhaps as a result of the Egyptian influence that geometry especially attracted the interest of Greek mathematicians, who did not just make improvements but took the subject to a whole new level. As was the case with so many other aspects of knowledge, the Greeks applied their speculative and rigorous mental attitude to mathematics, imbuing it with a scientific dimension, in the modern sense of the word. Mathematical properties were not proven in Egypt; they started with examples and derived properties from observation. The Greeks, in contrast, searched for the reason behind each phenomenon and wanted to prove properties based on axioms. The Egyptians only sought solutions to practical problems; the Greeks loved knowledge in its own right and studied mathematics without a primary regard for its use.

The Babylonian influence, on the other hand, is clear in Greek astronomy. It is thanks to the Greeks that the Babylonian sexagesimal system was handed down to us in the present day. In fact, the terms 'minute' and 'second' come from their use by the Greeks, albeit via their translation into Latin. They appeared for the first time in a text from the 13th century AD that made reference to a sixtieth part as the 'first smallest part;' the sixtieth parts of a sixtieth part as the 'second smallest part'. The Latin translation is *pars minuta prima*, *pars minuta secunda*, etc. Hence the familiar modern terms: 'minute' and 'second'. However, in reality, the journey by which these terms reached the present day was somewhat more complicated. The Latin text from the 13th century was not a direct translation of the Greek, but an Arabic adaptation of the original Greek. This reminds us once again, that the legacy of ancient Greece has reached the West largely thanks to the Arabs, who curated it for centuries.

29

The Greek numbering system was developed around 500 BC in Ionia and has some similarities to the Egyptian hieratic system. For example, it had a symbol for each number from 1 to 9, another symbol for each ten from 10 to 90, and another symbol for each hundred from 100 to 900. The symbols correspond to the Greek letters and to three Phoenician letters: *digamma* (to represent 6), *koppa* (to represent 90) and *sampi* (to represent 900).

The Greek symbols made it possible to represent any number between 1 and 999. Units preceded by a comma were used to represent thousands; hence the expression ',α' corresponds to 1,000 ',β' corresponds to 2,000, etc. Like the Egyptian system, this is an additive system of numerical notation, such that the number ρκε corresponds to 125, since ρκε = ρ + κ + ε = 100 + 20 + 5. The following table shows the letters that represent the basic numerical values:

| 1 | α | 10 | ι | 100 | ρ | 1,000 | ,α |
|---|---|----|---|-----|---|-------|----|
| 2 | β | 20 | κ | 200 | σ | 2,000 | ,β |
| 3 | γ | 30 | λ | 300 | τ | 3,000 | ,γ |
| 4 | δ | 40 | μ | 400 | υ | 4,000 | ,δ |
| 5 | ε | 50 | ν | 500 | φ | 5,000 | ,ε |
| 6 | ς | 60 | ξ | 600 | χ | 6,000 | ,ς |
| 7 | ζ | 70 | ο | 700 | ψ | 7,000 | ,ζ |
| 8 | η | 80 | π | 800 | ϖ | 8,000 | ,η |
| 9 | θ | 90 | φ | 900 | ϡ | 9,000 | ,θ |

The letter M was used, preceded by the letters corresponding to the corresponding multiple, to represent multiples of 10,000 up to 99,990,000. The letter M represented 10,000 and was derived from the Greek word for myriad, *murios* (μυριαδος), which meant 'one hundred times one hundred'. The number symbols were written either before or above the M, thus

$$\varpi o\alpha M = \overset{\varpi o\alpha}{M} = 8{,}710{,}000,$$

or

$$\varpi o\alpha M{,}\delta\rho o\delta = \overset{\varpi o\alpha}{M}{,}\delta\rho o\delta = 8{,}714{,}174.$$

To represent larger values, $10{,}000^2$ was represented as MM (a myriad myriad).

Order did not matter in Egyptian notation, but in Greek notation the numbers were written in line with Western reading conventions. The number began on the left with the largest values. This made it possible to eliminate commas if they were not required to understand meaning. However, as numbers were represented using letters, it was often necessary to distinguish between numbers and text. To do so, the Greeks added a mark at the end of the number or a bar above it. Hence, the number 871 was written as:

$$\varpi o \alpha ',$$

or

$$\overline{\varpi o \alpha}.$$

The development of calculation using this notation on paper is complicated to say the least. It is believed that the Greeks made use of the abacus for arithmetic, although mathematicians must have also used the symbols.

The Greek method of multiplication differs to what is used today. Nowadays, we take the digits of the second operand and consider the products of each digit of the first operand. On the other hand, the Greeks multiplied the digits of the first operand, by each of the digits of the second. However, since the position of digits had a value relative to their position (in the expression $\pi\delta$, the value of $\pi$ would always be 80 and not 8), the multiplication gave the value of the product as a direct result.

To calculate the product $24 \cdot 53$ (in Greek, $\kappa\delta$ multiplied by $\nu\gamma$) it is necessary to begin with $\kappa$, which is 20), which should be multiplied by the digits of 53; i.e. $20 \cdot \nu$ and $20 \cdot \gamma$ (in our current notation, $20 \cdot 50$, and $20 \cdot 3$). It is then necessary to do the same with the second digit of the first, multiplying: $\delta$, which is 4, by $\nu$ and then $\delta$ by $\gamma$ (in our current notation, $4 \cdot 50$ and $4 \cdot 3$). These partial results are then added together, to give the following in our current notation:

$$24 \cdot 53 = (20 + 4) \cdot (50 + 3) = 20 \cdot 50 + 20 \cdot 3 + 4 \cdot 50 + 4 \cdot 3 = 1{,}272.$$

Graphically, Greek notation is represented as follows:

| | | | |
|---|---|---|---|
| κ δ | | 24 | |
| ν γ | | 53 | |
| —— | | —— | |
| ,α ξ | | 1,000 | 60 |
| σ ιβ | | 200 | 12 |
| —— | | —— | |
| ,ασ οβ | | 1,200 | 72 = 1,272 |

The use of 27 symbols makes it difficult to handle partial products, since the corresponding multiplication table would require a total of $27 \cdot 27 = 729$ possible results. This was believed to be the reason the abacus played a fundamental role as a tool for calculation. The Greek abacus was a board with a number of columns in which stones or counters were placed. Each column had a value corresponding to the powers of 10, and there were other columns for fractions.
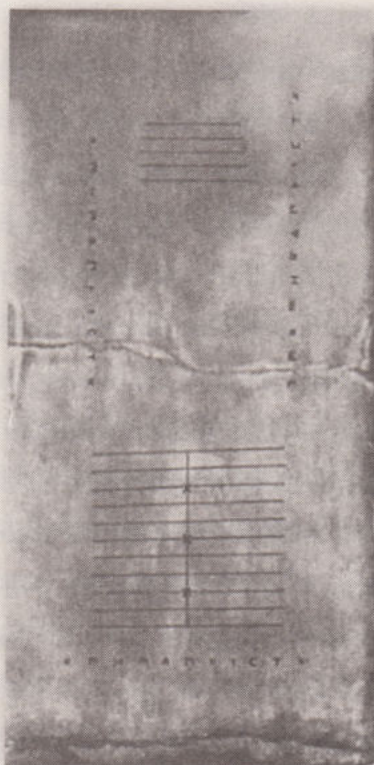


*The Salamis Tablet is a marble abacus that was discovered on that Greek island in 1846.*

It has been possible to study these tablets directly since some, such as the Salamis Tablet, have been discovered with information about the values in the columns. In this example, each column represented a certain quantity of Greek coins. The long

columns represented (from right to left) 1, 10, 100, 1,000 and 10,000 drachma, and then 1, 10, 100, 1,000 and 10,000 talents (one talent was equivalent to 6,000 drachma). The short columns represented fractions of the unit. The fractions of the drachma are the obol (one obol is 1/6 of a drachma), the hemiobolion, the tetartemorion and the chalkus (one chalkus is 1/8 obol). On the tablet, the pieces placed below the line represent one unit and those placed above the line represent five units. Hence, the following image represents the number 302,158 + 2 obols + 1/2 obols + 1 chalkus.
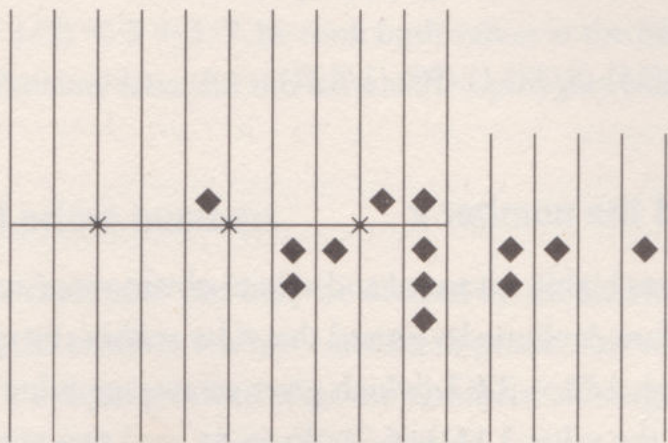
Addition on the abacus worked by adding the pieces depending on their position. When there were 5 units below the line, these were substituted for one unit above the line, and similarly, 2 units above the line were replaced by one unit on the next line. The operator had to remember that, 6,000 drachma should be exchanged for one talent, and fractions should be handled in a similar way, hence six obols were equivalent to one drachma.

| KANONION ΤΩΝ ΕΝ ΚΥΚΛΩ ΕΥΘΕΙΩΝ. | | | | | | | |
|---|---|---|---|---|---|---|---|
| ΠΕΡΙΦΕ-ΡΕΙΩΝ. | ΕΥΘΕΙΩΝ. | | | ΕΞΗΚΟΤΙΩΝ. | | | |
| Μοιρῶν. | Μ. | Π. | Δ. | Μ. | Π. | Δ. | Τ. |
| ō  ς″ | ō | λα | κε | ō | α | β | ν |
| α  ō″ | α | β | ν | ō | α | β | ν |
| α  ς″ | α | λδ | ιε | ō | α | β | ν |

*A table from the Almagest, a Greek work of astronomy written by Ptolemy in the 2nd century AD, which made use of fractions.*

Like the Babylonians, the Greeks were aware of sexagesimal fractions, as shown by Ptolemy in his *Almagest*, however they used the Egyptian system for mathematics. In his commentaries on the work of Archimedes, Eutocius of Ascalon uses

$$\overline{\alpha\varpi\lambda\eta}\ \theta'\iota\alpha'$$

to represent 1,838 (1/9) (1/11), and

$$\overset{\iota\alpha\,\iota\alpha}{\overline{\beta\eta\eta\varphi\theta'\rho\kappa\alpha'}}$$

to denote 2 (8/11) (8/11) (1/99) (1/121).

## The Greeks and the number π

Greek geometry was highly advanced and able to obtain more accurate approximations of π than before. Archimedes proved that π lay within the range 3 + 10/71 = 223/71 < π < 3 + 1/7 = 22/7 (which gives an average value of 3.141851), and Ptolemy obtained the value 3.141666. To do so, he used two regular polygons, one inscribed (drawn inside the circumference) and another circumscribed (outside the circumference), and compared their corresponding perimeters.



*Engravings dedicated to Archimedes (left) and Ptolemy.*
*Both men derived approximations of the number π.*

Archimedes began from the fact that a hexagon inscribed in a circumference with a radius of 1 has a perimeter of 6 and a circumference (the circle formed around it) of, $4 \cdot \sqrt{3}$. Hence, $\pi$ must be between 3 and $2 \cdot \sqrt{3}$. He then used the fact that the square root of 3 satisfies the following inequality $265/153 < \sqrt{3} < 1{,}351/780$. He continued the process using regular polygons with more sides. Starting with a hexagon (a 6-sided polygon), Archimedes repeatedly doubled the number of sides, first considering a polygon with 12 sides, and then 24, 48 and up to 96 sides. Using the polygon with 96 sides, he derived the approximation $6{,}336 / (2{,}017 + 1/4) < \pi < 14{,}688 / (4{,}673 + 1/2)$. Since $3 + 10/71 < 6{,}336/ (2{,}017 + 1/4) < \pi < 14{,}688 / (4{,}673 + 1/2) < 3 + 1/7$, he took both values as the limits of the interval that contained $\pi$. Ptolemy later did the same with a polygon with 360 sides.

## The Greeks and prime numbers

Prime numbers are those that can only be divided by themselves and the number one. By definition, the number 1 is not considered prime. Any natural number can be factorised into a unique product of prime numbers, although permutations or ordering can, of course, change. Hence, for example:

$$120 = 5 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 2 \cdot 5 \cdot 2 \cdot 2 \cdot 3.$$

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### PRIME NUMBERS BELOW ONE THOUSAND

Below is a list of the prime numbers under 1,000, in case a curious reader wishes to check their famous properties but does not wish to take the trouble of calculating them individually:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997.

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The Greeks studied prime numbers in depth. They defined the concept of a prime number and proved their most important properties. It is believed the Egyptians were also aware of the concept of prime numbers, however no results for the numbers are known prior to the Greeks.

In 300 BC, Euclid, who was a mathematician in Alexandria during the reign of Ptolemy I (323–283 BC), an era that witnessed another fortuitous connection between Egypt and Greece, discovered what is perhaps the most amazing and relevant property of prime numbers. He described the property *Elements of Geometry*, one of the seminal texts of mathematics that establishes the foundations of geometry, which future generations would build on – and dig into – over the course of the next two millennia. In Book IX of the *Elements*, Proposition 20 proves there is an infinite number of prime numbers.

Euclid's proof works by taking a set of prime numbers $S = \{p_1, p_2..., p_n\}$, and proving the number $N = p_1 \cdot p_2 \cdot ... \cdot p_n + 1$ is not divisible by $p_1$, because division by $p_1$ gives remainder 1. Likewise, $N$ is not divisible by $p_2 ..., p_n$, because division of $N$ by $p_2 ..., p_n$ gives remainder 1. Hence, $N$ is either prime or $N$ is a number obtained from the product of two primes, not included in $S$. Hence, $S$ cannot be a full set of primes. Since the selection of $S$ is arbitrary, there can be no finite list of prime numbers and, consequently, the list of primes is infinite.



*Detail from Raphael's* The School of Athens, *in which Euclid appears teaching geometry.*

# Rome

Mathematics and its notation were not as powerful in Rome as in Greece or Babylon and nor did the subject evolve to the same extent. The capital of the Latin world, so fertile in other respects, did not produce any distinguished mathematicians. During Roman times, the most important moments in the development of mathematics did not take place in the heart of the empire, but occurred on the periphery, in areas of Greek influence, where the tradition of Greek mathematics continued. In fact, it is widely believed that Roman mathematics came from a completely different tradition, unrelated to that of the Greeks and Babylonians but based on the Etruscan numbering system and mathematics. The key authors from this period, mathematicians from the Greek tradition, were Ptolemy, with the aforementioned *Almagest*, and Diophantus and Pappus of Alexandria. Diophantus wrote a work entitled *Arithmetic*, and Pappus wrote a collection of eight books of commentary on the subject.

Cicero himself recognised the limitations of Roman mathematics in his *Tusculan Disputations*, in which he affirmed that:

"Geometry was in high esteem with them, therefore none were more honourable than mathematicians. But we have confined this art to bare measuring and calculating." (*Tusculan Disputations*, I, 5).
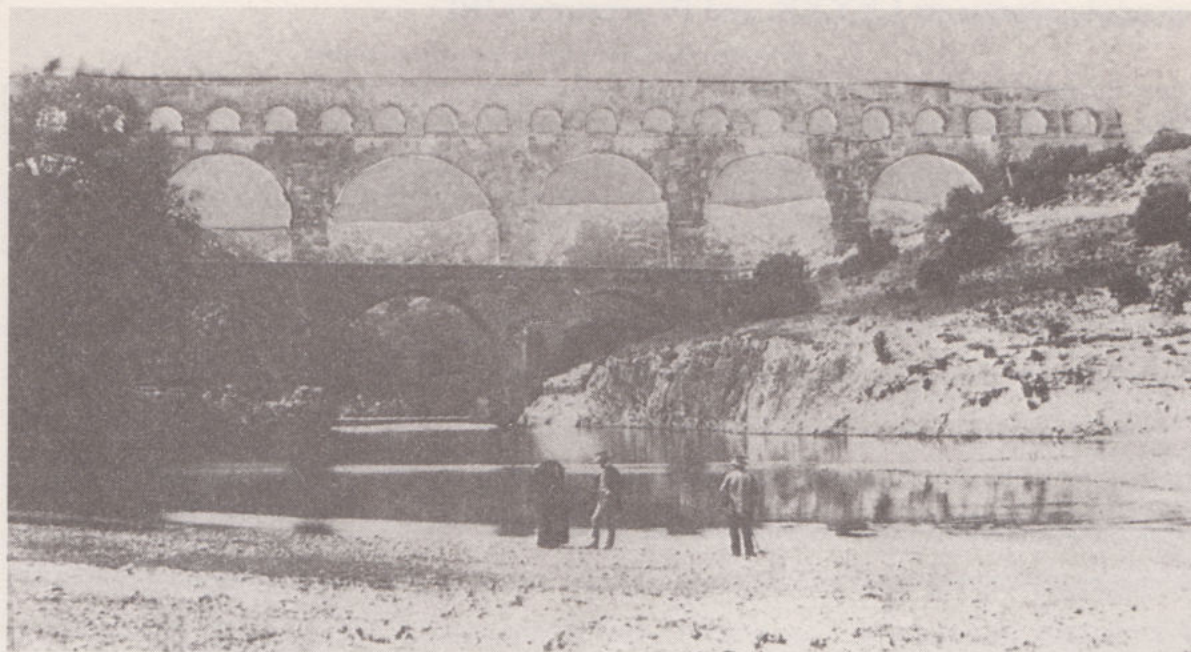
The Pont du Gard, photographed by Édouard Baldus in the second half of the 19th century. This aqueduct, which is also a viaduct, was built by Roman engineers who employed ancient mathematical knowledge in their work.

However, it is important not to lose perspective. It may be the case that the Romans did not make significant advances in mathematics and calculation, leaving the Greeks unbeaten, but there can be no doubt that they led the way in ancient technology and engineering, for which they made masterful use of mathematics. Many of their magnificent works of engineering and architecture have survived the passing of time and their admirable solutions, together with the mathematical ingenuity with which they were designed, have been handed down to us. Consequently, the Romans produced a vast range of texts on construction techniques, the most famous of which was by Vitruvius.

The original Roman numerical notation is still very familiar to us, since it currently has a wide variety of uses:

| 1 | 5 | 10 | 50 | 100 | 500 | 1,000 | 5,000 | 10,000 | 50,000 | 100,000 | 500,000 |
|---|---|----|----|-----|-----|-------|-------|--------|--------|---------|---------|
| I | V | X | L | C | I) | (I) | I)) | ((I)) | I))) | (((I))) | I)))) |

Later on, the printing press simplified | ) with D and ( | ) with M, and also created new notation for expressing larger values. A bar above the number multiplied it by 1,000, and vertical bars on either side multiplied it by 100. For example, |LV| represented 5,500. They also introduced values to the left, for subtraction. XC, for example, represents LXXXX, and IV represents IIII.

## THE VITRUVIAN FRAME

The famous architect and writer, Vitruvius (80–15 BC) served in the legions of Julius Caesar, from whom he received direct orders. The Roman endowed cultural history with one of his great works, *On Architecture*. The 10 volumes cover a broad range of aspects of the discipline from the Roman perspective, ranging from construction elements, such as machines or materials, all the way through to urban and country planning.

*An edition of Vitruvius' On Architecture published in 1567.*

M·VITRVVII POLLIONIS
DE ARCHITECTVRA
LIBRI DECEM,

CVM COMMENTARIIS
DANIELIS BARBARI,
ELECTI PATRIARCHAE
AQVILEIENSIS:

MVLTIS AEDIFICIORVM, HOROLOGIORVM,
ET MACHINARVM DESCRIPTIONIBVS,

CVM PRIVILEGIIS.

VENETIIS,
Apud Franciscum Franciscium Senensem, & Ioan. Crugher Germanum.
M. D. LXVII.
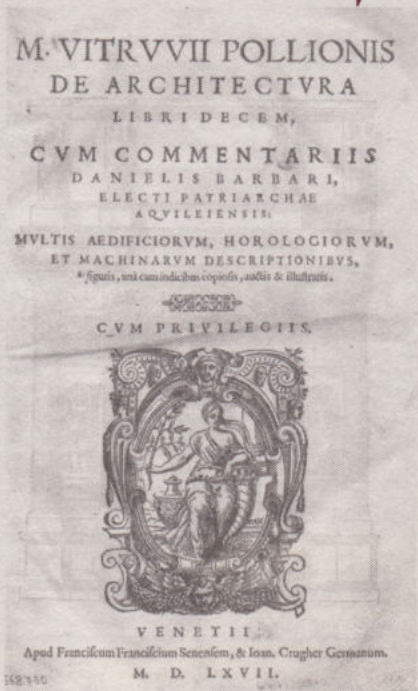
However, using this numbering system was complicated. It was most likely that an abacus or calculating table was used for calculation, and that the numerals were only used to record the values and the result. Roman calculating tables worked in a similar way to those of the Greeks. The table had a number of lines: the pieces positioned above the line corresponded to units, and those positioned on it corresponded to 5 units. The table was divided into two parts: the right-hand side represented the number to be added and the left, the result.

By means of example, in the diagram below the position on the left represents the number 2,907, and the position on the right, 43. Transferring all the stones to the left, and replacing the surpluses – five stones on a line or two between the lines result in one stone moving to the next position above – gave the result of the addition: 2,950.
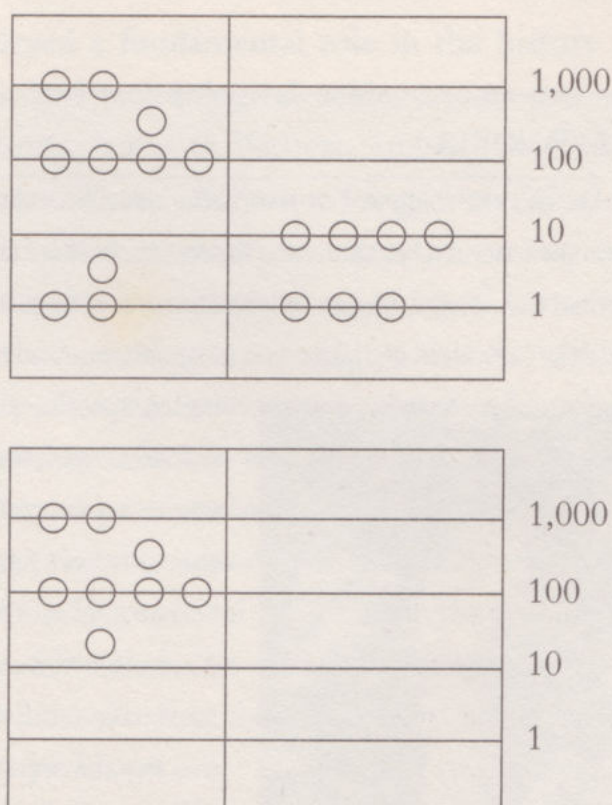
In addition to the calculating table, the Romans also made use of a metal or wooden tablet with grooves, on which they placed small stones to denote numbers. Although it may seem unlikely, these stones represent Rome's great contribution to mathematics. The Latin word for 'stone,' which was *calx*, and its diminutive *calculus*, which meant 'small stone' or 'pebble,' forms the root of the modern word to 'calculate'.

# Mathematics in Alexandria

The great advances made during eight centuries of Roman dominance were of Greek parenthood. Alexandria, a Greek city situated in Egypt, with its museum and library, was the world's most important centre of learning, and some of the most important Greek mathematicians of the late Roman Empire lived there.

The previously mentioned Pappus of Alexandria lived at the start of the 4th century and attempted to recover Greek mathematics by compiling and writing commentaries on earlier texts. His work extended previous discoveries with more detailed proofs that allowed readers to achieve a better understanding of the ancient works. Unfortunately, he was not successful in his aim. Few mathematicians of renown arose as a result of his efforts.

## HYPATIA OF ALEXANDRIA

Hypatia (c. 370–415 AD) was the daughter of the mathematician and philosopher Theon of Alexandria, from whom she inherited her talent and interest for intellectual endeavours. She was a pagan at the time when the newly Christian Roman Empire had begun to persecute them. In spite of this, the great mathematician enjoyed some independence and even taught the future bishop Synesius of Cyrene as one of her students. However, in 415, she became embroiled in a political battle between the Christian patriarch Cyril and the Roman prefect Orestes, to whom she was both friend and counsellor. In order to damage Orestes, a rumour was spread that Hypatia practised witchcraft. A mob then brutally murdered her.

*Hypatia of Alexandria, with a white tunic, appears in this detail from Rafael's* The School of Athens.

One of the few brilliant figures who did appear after Pappus was the famous Hypatia, also from Alexandria, whose mathematical texts include commentaries on the work of Apollonius of Perga (*On Conical Sections*) and Diophantus's *Arithmetic*, in addition to parts included in her own father's commentaries on Ptolemy's *Almagest*. Her assassination in 415 AD, which transformed her into the great martyr of science and feminism, together with the destruction of the Museum of Alexandria and its important library, which occurred at some point between the 4th and 7th century, saw the Greek mathematical tradition go up in flames and buried it below the rubble of barbarity, from where it was unearthed by Arab scholars.

## China

Mathematics has played a fundamental role in the history of China, a history filled with scientific and technological achievements that were often ahead of their time, when compared with Western endeavours. From the times of the Han dynasty (206 BC–220 AD), selection for governmental posts was based on strict examinations and not, as might be expected, on family relations. Entrance examinations placed particular emphasis on classical Chinese literature, but also included mathematical problems, something that set this culture apart from others. Despite many discontinuities, this model is still applied in modern times. The exam system did not favour mathematical creativity, and in general the Chinese relied on lists of problems and solutions to be memorised. As is logical, they shared the vision of the sciences of the Babylonians and Egyptians, who gave them as a primarily practical role rather than intellectual. Nevertheless that did not stop a culture so diverse and long lasting from seeking out new mathematical knowledge to find new and more efficient methods of solving increasingly complex problems.

The most important ancient Chinese mathematical text is *The Nine Chapters on the Mathematical Art*. Generations of Chinese mathematicians studied *Jiǔzhāng suànshù* as it is known in its original language. They also drew upon the commentaries and annotations made by Liu Hui in the 3rd century BC. In addition to this, in 1983 a tomb was discovered from 186 BC with 190 strips of bamboo recording mathematical texts. Each strip is 30 centimetres long and 6 or 7 millimetres wide and together they make up a text of around 7,000 characters. The strips were originally joined together and then rolled up, however they were discovered out of sequence because their links had decayed. Their reconstruction caused quite a headache for the experts.
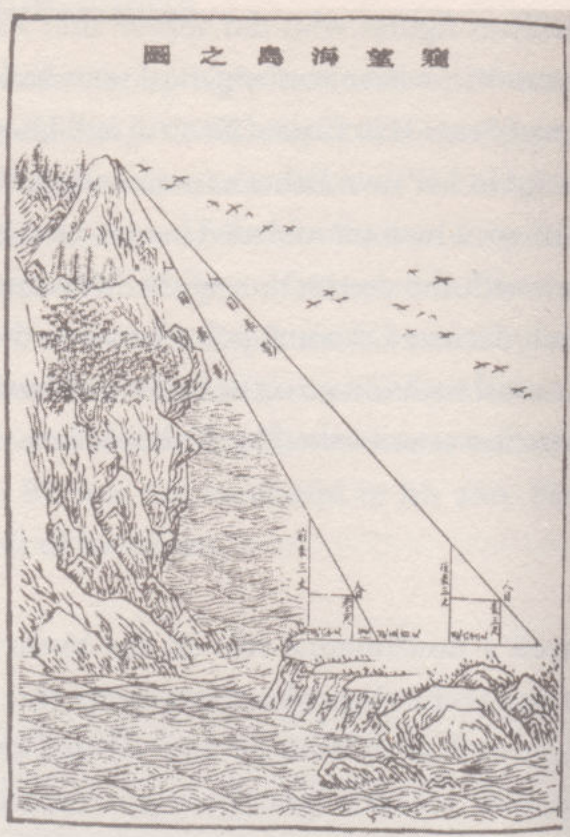
*An 18th-century reproduction of a problem by the mathematician Liu Hui, which considers how to measure the height of an island peak.*

Once reordered, the great importance of the text was revealed. It includes various types of problems, based around applications such as figuring tax and calculating volumes. However despite being a primarily practical document, its problems and solutions do reveal interesting techniques, such as the false position rule or algorithms for extracting square roots. In addition, many of the problems are presented as enjoyable and frequently allegorical stories.

Although the next sections deal with the parts of Chinese mathematics that are most relevant to this book, such as numbers and calculation, it is worth noting that the

## A PROBLEM FROM *THE NINE CHAPTERS*

The problem set out on strips 34 and 35 of *The Nine Chapters on the Mathematical Art* is typical of the type of problems described in the text as a whole: A fox, a mountain cat and a dog pass through a customs post and must pay 111 coins. The dog says to the cat, and the cat to the fox, "your fur is worth twice as much as mine; you should pay double the tax." How much should each animal pay?

Chinese mathematicians mentioned made many other important discoveries, which are perhaps beyond the scope of this book, but that were nonetheless fundamental to the history of mathematics. These include methods for solving equations and problems related to congruences.

## Numerals and the calculation system in China

The oldest form of calculating used by the Chinese dates back to the 4th century BC. It consisted of a number of small sticks or bars used for counting, which were referred to as (算) or chou (筹). As time passed, the system was replaced by the abacus. The sticks represented the numbers from 1 to 9 and made use of two possible layouts. The first set used the vertical position of the small sticks, as shown in the following representation, which shows the numbers from 1 to 9, from left to right:
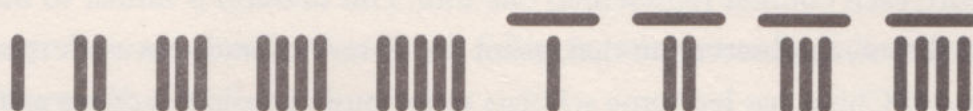
The second set uses the horizontal position, as shown below, which again shows the numbers from 1 to 9:
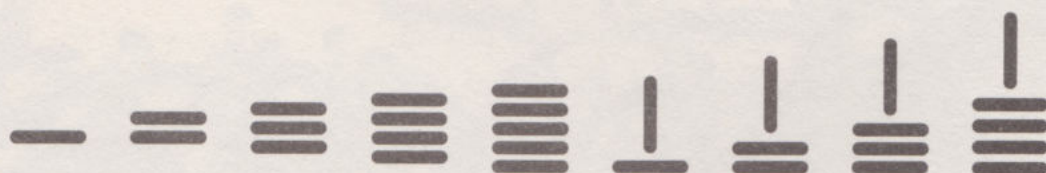
This numbering system was used in conjunction with tablets to represent numbers depending on the position of the stick symbols. For example, the representation of 4,508 on the tablet would be as follows:

As can be seen, the two series were alternated to represent each digit. The vertical series was used for units, hundreds, etc. while the horizontal series for tens, thousands, etc. When one of the digits was zero, the corresponding position was left empty, as can be seen above.

The same system was also used to represent negative numbers. The different number types were distinguished by the colour of the sticks: Positive numbers were represented using red sticks and negative numbers using black ones.

Calculation was carried out on the same tablet using the same sticks. Addition and subtraction were achieved by adding or removing sticks from the tablet. There were also methods for carrying out multiplication and division, and even for representing and calculating other algebraic operations such as polynomials.

It is not known when the system of calculating using sticks was introduced to Korea and Japan, but at least in the case of Japan, it is known that it was already being used in the times of Empress Suiko (593–628 AD) and was referred to as *sangi*.

The abacus was known in China from the 2nd century BC, and was referred to as the *suanpan*. The Chinese abacus was split into two parts: on the upper part, the counters represented five units (or tens, hundreds, etc. as applicable) and on the lower part, each counter represented one unit. This division is similar to that of the Roman abacus, an observation that, given the history of trade between the Roman Empire and China, has led some scholars to seriously consider a direct relationship between one version of the abacus and the other.
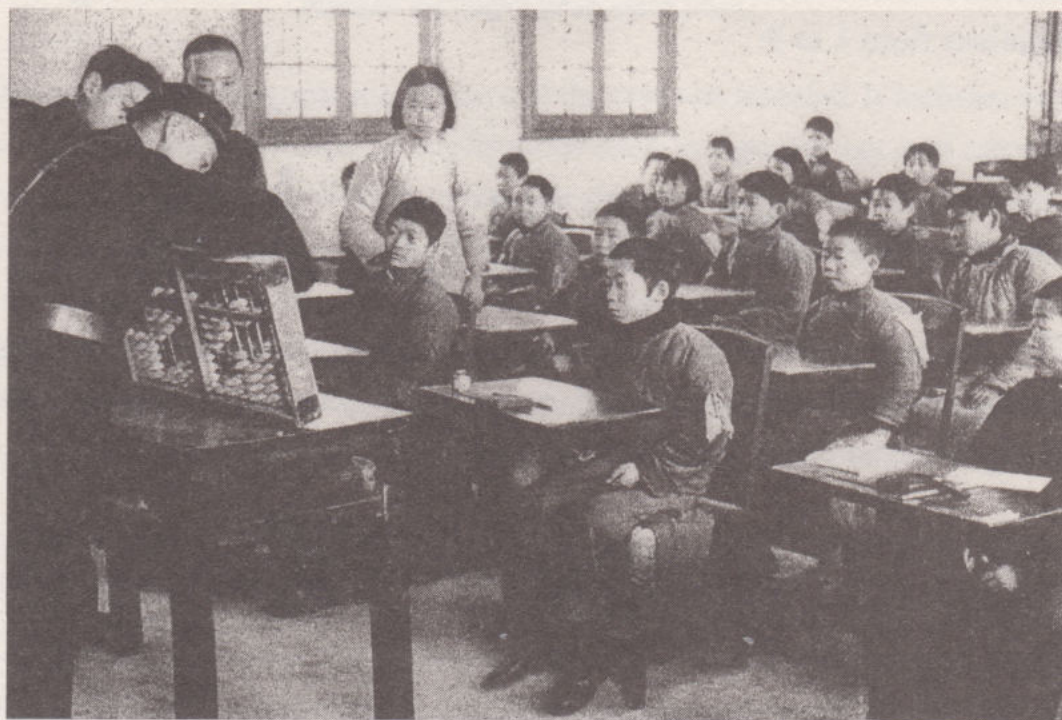
*Students being taught how to use the abacus in a school in the Chinese district of Zhenjiang in a photograph taken in 1938.*

The Chinese abacus was introduced into Japan around the 16th century and was named the *soroban*. It reached Japan as a result of trade, but its dissemination was not easy and it met with considerable reticence. Many years would pass until it was accepted in schools and used for carrying out advanced mathematics. Japan's merchants were more open to using the *soroban* for commercial transactions, but more traditional techniques maintained their dominance when it came to pure mathematics.

The representation of numbers, both in China and Japan, the latter system being derived from its larger neighbour, makes use of nine ideograms to represent the digits from 1 to 9:

| 一 | 二 | 三 | 四 | 五 | 六 | 七 | 八 | 九 |
|----|----|----|----|----|----|----|----|----|
| 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |

The symbols are combined with the following ideograms to indicate tens, hundreds, thousands, etc.:

十 : Ten (10).

百 : Hundred ($10^2$).

千 : Thousand ($10^3$).

万 : Ten thousand ($10^4$).

億 : One hundred million ($10^8$).

Numbers are written using the symbols for 1 to 9, punctuated by the symbols for tens, hundreds, etc. For example, 10,563 would be represented as follows:

一万五百六十三.

This can be understood as:

一 (one), 万 (ten thousands), 五 (five), 百 (hundreds), 六 (six),
十 (tens), 三 (and three units).

It is interesting to note that, in contrast to the system used in the majority of European cultures, where the thousand ($10^3$) is of fundamental importance and provides the basis for building multiples, in this case it is the unit $10^4$ (ten thousands)

that is used for building multiples. As such, 132,000 is written as $13 \cdot (10^4) + 2,000$, which is written below using the ideograms:

十三万二千

## The number π in China

The Chinese developed algorithms to calculate the number π. The great mathematician Liu Hui, who lived around the year 300 AD during the reign of Wei, after the collapse of the Han Empire, was the first to devise a method for calculating the value of π. Prior to this, the scientist and inventor Zhang Heng (78–139), who designed an earthquake detector some 1,700 years before the first seismograph, had calculated the approximation of 3.1724. Other approximations of 3.162 (the root of 10) and 3.156 were also known. In the 3rd century, the astronomer Wan Fan, in the reign of Wu, used the latter value as the result of the fraction 142/45.

The first method used by Liu Hui to calculate an approximation of π worked by bisecting polygons. Using a 96-sided polygon, he obtained an approximation of π between 3.141024 and 3.142708, which led him to approximate π as 157/50, since he believed 3.14 was sufficiently precise.

*Chinese stamps in honour of the intellectuals Liu Hui (left) and Zhang Heng.*

Liu Hui's procedure began with an inscribed hexagon, with sides of length $L$. An iterative process was then applied, with the polygon's number of sides being doubled at each step. Hence, beginning with a hexagon, step two considered a dodecagon (12 sides), then a 24-sided polygon ($= 12 \cdot 2$) and a 48-sided polygon ($= 24 \cdot 2$), and so on. At each step, after considering the polygon with $N$ sides, he calculated its area and the length of the side of the polygon with $2N$ sides.

Let's represent the length of the side of the polygon with $2N$ sides by $l$. To do so, we apply Pythagoras' theorem: Given a right-angle triangle with a hypotenuse of length $h$ and two legs (short sides) with lengths $c_1$ and $c_2$, $h^2 = c_1^2 + c_2^2$.
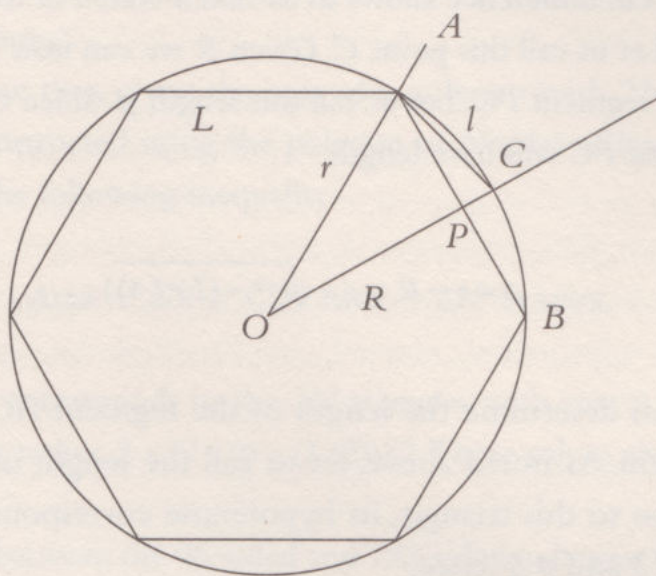


*A representation of the process for calculating the length* l *based on the length* L, *where* L *is the length of a side of the hexagon and* l *is the length of a side of the 12-sided dodecagon.* O *represents the centre of the circle;* A *and* B, *two vertices of the hexagon;* C, *the new vertex of the dodecagon, and* P, *the bisection point on the side of the hexagon that is half way between* A *and* B. *The length of the radius is* r, *and the length from the centre to P is* R.

In the diagram, the centre of the circumference is indicated by $O$, and the side in consideration (length $L$) has two end points $A$ and $B$. Note that $OAB$ defines an equilateral triangle. Hence, the procedure is as follows.

Step 0. Consider the polygon with $N = 6$ sides, with sides of length $L$ known.

Step 1. First divide the side $AB$ into two equal parts. The midpoint is indicated by $P$.

Step 2. Now calculate the length of the segment $OP$ and let $R$ be this length. To do so, apply Pythagoras' theorem to the triangle $OAP$. At this point,

we already know that the hypotenuse of the triangle is $r$, that one of the legs is $L/2$ and the other, which we wish to calculate, is $R$. Hence, by applying Pythagoras' theorem, we know the following holds: $r^2 = R^2 + (L/2)^2$. Hence $R^2 = r^2 - (L/2)^2$, and thus:

$$R = \sqrt{(r^2 - (L/2)^2)} = \sqrt{(r^2 - (L^2/4))}.$$

Step 3. Now consider the radius that passes through $P$. The extension of this radius to the circumference allows us to find a vertex of the polygon with $2N$ sides. Let us call this point $C$. Given $R$ we can now calculate the length of the segment $PC$. Let us call this length $\rho$. Since $OC$ has length $r$, the segment $PC$ will have length

$$\rho = r - R = r - \sqrt{(r^2 - (L^2/4))}.$$

Step 4. We can determine the length of the segment $AC$ using Pythagoras' theorem. As noted above, let us call the length of this segment $l$. In relation to this triangle, its hypotenuse corresponds to $l$ and the legs are $L/2$ and $\rho$. Hence,

$$l^2 = (L/2)^2 + \rho^2 = L^2/4 + \left(r - \sqrt{r^2 - (L^2/4)}\right)^2 = 2r^2 - 2r\sqrt{r^2 - (L^2/4)}.$$

Step 5. Solving the last equation for $l$, we can obtain the length of the side of the polygon with $2N$ sides:

$$l = \sqrt{2r^2 - 2r\sqrt{r^2 - (L^2/4)}}.$$

Step 6. The area of the polygon with $N$ sides can be calculated based on the triangle defined by the points $OAB$. Hence, the area of the polygon will be $N$ times the area of the triangle. The area of the triangle $OAB$ is the product of its base and height divided by 2. The base $AB$ has length $L$, and the height is the previously calculated value $R$. Hence, the area of

the polygon is

$$N \cdot \text{area of the triangle } OAB = N \cdot (L \cdot R)/2.$$

Step 7. Now repeat step 2, letting $N = 2N$, $L = l$.

To calculate the value of $\pi$, note that the area of the circumference is $\pi \cdot r^2$. Hence, for $r = 10$, the area is $\pi \cdot 100$.

If we begin with $r = 10$, which gives $L = 10$, the areas calculated will be those in the table presented below, using modern notation (Liu Hui used fractions to denote the calculations).

Liu Hui also saw that, given the area of a polygon with $2N$ sides of length $l$ (denoted as $C$), constructed using the polygon of $N$ sides of length $L$, the area of the circle satisfies the following inequality:

$$\text{Area} - 2N < C < \text{Area} - 2N + \text{extra}.$$

Here, the extra corresponds to the $2N$ triangles with area $\rho \cdot (L/2)/2$. Recall that $\rho = r - R$. Or rather, $2 \cdot N \cdot (\rho \cdot (L/2))/2$. These values are also given in the table below.

The difference between the 96-sided and 192-sided polygons is extremely small, and hence Liu Hui decided that $\pi = 3.14$ was a sufficient approximation.

| Number of sides | Length of side of polygon | Length of side of polygon with $2N$ sides | | Area of polygon | Difference between area of polygons with $N$ and $2N$ sides |
|---|---|---|---|---|---|
| $N$ | $L$ | $l$ | $R$ | $N \cdot (L \cdot R)/2$ | $2N \cdot (\rho \cdot (L/2))/2$ |
| 6 | 10 | 5.1763806 8 | 8.6602545 | 259.80765 | 40.192368 |
| 12 | 5.1763806 8 | 2.6105237 | 9.659258 | 299.99997 | 10.582865 |
| 24 | 2.6105237 | 1.3080626 | 9.914449 | 310.58282 | 2.6800032 |
| 48 | 1.3080626 | 0.65438163 | 9.978589 | 313.26285 | 0.67216444 |
| 96 | 0.65438163 | 0.32723463 | 9.994646 | 313.935 | 0.16816857 |
| 192 | 0.32723463 | 0.1636228 | 9.998661 | 314.10318 | 0.042062752 |
| 384 | 0.1636228 | 0.08181208 | 9.999665 | 314.14526 | 0.01051604 |

Liu Hui noted a certain connection between the successive extra quantities of these results. In particular, he saw that the ratio between one extra area and the following is approximately $1/4 = 0.25$. (These quotients are included in the next table.) This ratio makes it possible to obtain an approximation of the area of the polygon with 3,072 sides, and based on this, derive a more precise approximation of $\pi$.

As an example, consider the estimation given by Liu Hui's approximation of the polygon with 384 sides based on the last calculation (192 sides). In this case, the area of the 192-sided polygon is 314.10318 and the extra of this polygon with respect to the previous one is 0.16816857. Based on this, Liu Hui estimated the difference in areas between the 192-sided and the 384-sided polygon to be $0.16816857 \cdot (1/4)$ = 0.042042144. Hence, the area of the 384-sided polygon is:

$$314.10318 + 0.16816857 \cdot 0.25 = 314.14523.$$

We can see that the actual extra is 0.042062752 and the total area is 314.14526. Liu Hui proceeded using this same system until finding the area of the polygon with 3,072 sides, which allowed him to obtain his approximation of $\pi$ as $3927/1250 = 3.14159$.

| Number of sides | Length of side of polygon | Area of polygon | Difference between areas of polygons with $N$ and $2N$ sides | Quotient |
|---|---|---|---|---|
| $N$ | $L$ | $N \cdot (L \cdot R)/2$ | $2N \cdot (\rho \cdot (L/2))/2$ | Extra area($N$)/extra area ($N$ $N/2$) |
| 6 | 10 | 259.80765 | 40.192368 | |
| 12 | 5.1763806 8 | 299.99997 | 10.582865 | 0.26330534 |
| 24 | 2.6105237 | 310.58282 | 2.6800032 | 0.25323987 |
| 48 | 1.3080626 | 313.26285 | 0.67216444 | 0.25080734 |
| 96 | 0.65438163 | 313.935 | 0.16816857 | 0.25018963 |
| 192 | 0.32723463 | 314.10318 | 0.042062752 | 0.25012255 |
| 384 | 0.1636228 | 314.14526 | 0.01051604 | 0.25000837 |

The method was used again in 480 AD by the mathematician Zu Chongzhi (429–500), who applied it to a polygon with $12,288 = 3 \cdot 2^{12}$ sides to give the result that $\pi$ lay between the two values: $3.1415926 < \pi < 3.1415927$. This is most concisely expressed by the following fraction: $\pi \simeq 355/113$. For the next 900 years, this remained the most accurate approximation of that most fabled of numbers.

# Indian and Arabic mathematics: positional numbering

By tradition, history tells us that Indian mathematics was born in the 7th century AD when a cultural uniformity resulted from the widespread use of Sanskrit as the common language, stimulating advances and making it possible for them to be disseminated further afield. Nevertheless, in reality India was far from isolated from the rest of the world prior to this. There was intense contact with the ancient Greeks and then the Romans. It should not be forgotten that Alexander the Great extended his empire all the way to the Indus Valley, which is now the heartland of Pakistan.

Although Indian science placed a special emphasis on astronomy, it also made advances in mathematics, a fundamental tool for gleaning scientific knowledge. Curiously, the Indians did not share the Far Eastern concept of the sciences, and for them mathematics did not have to be always practical. Indian mathematicians were motivated by knowledge in its own right. In spite of this, their studies did not show excessive zeal when it came to providing formal proofs of methods or procedures. It is believed that Indian mathematicians had to justify their discoveries, although few of the corresponding proofs have been preserved.

The Indians studied trigonometry in depth, above all for use in carrying out astronomical calculations, although also for its use in indeterminate equations, algebra and combinatorics. In fact, the concept and the word *sine* come from a 5th-century astronomical work named *Paitāmahasiddhānta*.

## SINE

How did the word 'sine' come to be applied to a trigonometric concept? It all began with an astronomical treatise from India, entitled *Paitāmahasiddhānta*, which contained a table of jyā-ardha, or 'half chords', used for astronomical calculations. The term reappears in *Āryabhaṭīya*, the great work of the Hindu mathematician Āryabhaṭa, who shortened it to *jyā* o *jīvā*. The Arabs transcribed it into their language as *jiba*. However since Arabic does not use vowels, it appeared abbreviated to *jb* in texts. A subsequent reading, either a misunderstanding or perhaps ill-intentioned, interpreted *jb* as *jaib*, which meant 'bosom' or 'breast', and Latin translators interpreted it literally, as *sinus*, which means 'breast,' 'fold in the toga,' and 'bay' or more latterly simply 'curve'. It has since been anglicised into sine.

There is no doubt that the most significant contribution of Indian mathematics to science as a whole is the numeral system we now refer to as Arabic, although the Arabs themselves took the system from the Indians. The numbering system comes from the writing system used in the times of King Aśoka (272–231 BC): the Brahmi writing system the texts of which transcribe the ancient Prakrit language and also include symbols for numbers. However, the figures used for the digits underwent a long series of modifications on their journey to the West, and as such, the forms used today barely correspond to those first representations. In their current guise, the numbers are the version of those ancient Prakrit numerals that reached North Africa after a number of alterations, and spread throughout Europe during the Middle Ages.

| क | ख | ग | घ | ङ |
|---|---|---|---|---|
| ka = 1 | kha = 2 | ga = 3 | gha = 4 | ṅa = 5 |
| च | छ | ज | झ | ञ |
| cha = 6 | chha = 7 | ja = 8 | jha = 9 | ña = 10 |
| ट | ठ | ड | ढ | ण |
| ṭa = 11 | ṭha = 12 | ḍa = 13 | ḍha = 14 | ṇa = 15 |
| त | थ | द | ध | न |
| ta = 16 | tha = 17 | da = 18 | dha = 19 | na = 20 |
| प | फ | ब | भ | म |
| pa = 21 | pha = 22 | ba = 23 | bha = 24 | ma = 25 |
| य | र | ल | व | |
| ya = 30 | ra = 40 | la = 50 | va = 60 | |
| श | ष | स | | |
| śha = 70 | sha = 80 | sa = 90 | | |
| ह | | | | |
| ha = 100 | | | | |

*Fragment of Indian alphabetical numbering as described by the mathematician Āryabhaṭa (source: Georges Ifrah:* The Universal History of Numbers*).*

The positional system also has its roots in India. To start with, the ancient Indians wrote numbers using symbols for 1 to 9, and then another set of symbols to represent tens from 10 to 90. Multiples of 100, 1,000, etc. were built up by representing the units multiplied by the symbols for 100, 1000, etc. The notation was subsequently simplified, bequeathing history positional notation, which did not need more symbols than 0 to 9. There is controversy regarding the exact date of the transformation, but the majority of evidence suggests this was probably around the year 600 AD. At the latest, a text from Syria dated to 662 was already discussing Indian notation.

One theory suggests that perhaps the system arose on the Chinese border, where the use of the abacus created a requirement for an easier system for representing accounts using that calculation tool. The origins of positional numbering may be related to the use of the point to mark empty spaces on the abacus, a practice that is documented in a text from the 17th century, discovered in 1881 in Bakhshālī in the north east of India. The real revolution occurred when the point was converted into a zero. The zero is included as a numeral from the year 628 AD, when Brahmagupta defines it in his book *Brahmasphutasiddhanta* (*The Opening of the Universe*) as the result of subtracting a number from itself.

Regardless, by the year 870, the positional system was already established in India. From there it reached Baghdad, from where it would later spread throughout the entire Islamic empire and beyond. China made use of positional notation with its own characters from the Ming Dynasty onwards (1368–1644). Chinese characters were only replaced by the Arabic numeral system for mathematical writing in the 20th century.

The oldest Arabic book that has been preserved and makes use of Arabic numbering with positional notation is *Principles of Hindu Reckoning* by Kūshyār ibn Labbān, known in Arabic as *Kitāb fī usūl hisāb al-hind*. The work is not only significant

## THE WORD ZERO

The etymology of the words 'zero' comes from the Arabic word *sifr*, a transformation of the Indian word *sunya* that originally meant 'empty.' The word "zero" comes from the Latin word used by Fibonacci in his *Book of Calculation* (*Liber Abaci*), a text largely responsible for popularising Arabic numbers in Europe. Fibonacci writes of *zephyrum*, which means 'western wind' in Latin and Greek, possibly on account of its similarity to the Arabic word *safira*, which means 'to be empty' and is obviously related to *sifr*, 'empty'.

## KŪSHYĀR IBN LABBĀN

The Persian astronomer and mathematician Kūshyār ibn Labbān (971–1029) was born in Jīlān, to the south of the Caspian Sea. Although his most famous text is his renowned treatise on arithmetic, *Principles of Hindu Reckoning*, his written work is vast and contains books and collections of tables that have been handed down through generations of the Islamic scientific tradition. He was the teacher of the algorist, al-Nasawī. In his treatise on arithmetic, he presents Arabic numbering and defines its main operations: addition, subtraction, division by two, multiplication, division, the square root and the cubic root.

in terms of the numeral system it uses, but also on account of the originality of the mathematics. Zero appears throughout the text, referred to as *sifr*, and is used like any other number.

Up to this point, many texts written in Arabic were translations of Greek texts. However, in the 10th and 11th centuries this tendency would change dramatically. Around the turn of the millennium, when Kūshyār ibn Labbān was writing his works, texts with new and important mathematical results abounded in the Islamic literature. In fact, it was Muslim scholars who incorporated fractions into the positional notation system, which until then had only been developed for whole numbers.

## Calculating the number π in India

The Indians were also seduced by the mystery of the number π. The founder of the Kerala School of astronomy and mathematics, Madhava of Sangamagrama (1350–1425), discovered, among other things, infinite series for the trigonometric sine and cosine functions, which he used to define the number π by means of a decomposition of the arctangent function. He expressed π in the following way:

$$\pi / 4 = 1 - 1/3 + 1/5 - 1/7 + ... + (-1)^n/(2n+1) + ...$$

Moreover, he also perceived the error of calculating π using only *n* terms from the series. The calculation of the error from truncating the series implies considerable knowledge of the series itself. The arctangent series was later rediscovered by James Gregory and used by Gottfried Leibniz to calculate π using the expression

above. The formula for doing so is normally known as the Leibniz or Gregory-Leibniz series, and has only more recently been credited to its true inventor, with the modern name Madhava–Leibniz.

The expression used for the arctangent is as follows:

$$\arctan x = x - (x^3) / 3 + (x^5) / 5 - (x^7) / 7 + \dots$$

Nevertheless, this series is highly inefficient for the purpose of calculating the number $\pi$. The problem is that 10,000 mathematical operations are required for a calculation correct to 10 decimal places!

# Chapter 2

# Medieval Europe

During the first centuries of the Middle Ages, teaching in Europe built on the texts and authority of late Roman authors, such as Boëthius. Mediaeval universities educated their students in line with the model developed in the 5th century by the enigmatic lawyer Martianus Capella, author of *De Nuptiis Philologiae et Mercurio* (*The Marriage of Philology and Mercury*), a work also known as *De Septem Disciplinis* (*The Seven Disciplines*), in which he presented the classic division of knowledge into *trivium* and *quadrivium*, the first curriculum of further education.

The weight of the Roman cultural legacy extended to calculations. Roman numerals continued to be used for these. The slow process by which Arabic numerals were introduced, one full of controversy and bitter disputes, occupied a large part of this period. However, the Middle Ages did see important innovations that would have decisive consequences in the future, as was the case, for example with the system of logic developed by Ramon Llull, which heavily influenced the work of Leibniz in the 17th century.

## Boëthius and rithmomachia

Rithmomachia was a game that enjoyed considerable popularity in the Middle Ages. It was in some ways similar to chess and which and was developed in the second half of the 11th century in the monasteries of southern Germany. It reached its peak in the 16th century before making a steady decline and disappearing completely. Although it is just a game, studying the development of rithmomachia is of particular interest to historians of the sciences, since its rise and fall run parallel to progress in the mathematics of the era.

The dominant mediaeval work of mathematics is Boëthius' *Arithmetic*, the Latin title of which is *De Institutione Arithmeticae*. Its structure was extremely different to the mathematical works of our time. In many ways when seen through modern eyes it looks like a step backwards from the work of the Greeks. The work reflects upon the relationships between numbers and, above all, proportions, and defines a vast set of concepts, emulating Euclid's *Elements*. However, it does not include the ideas

# TRIVIUM AND QUADRIVIUM

The concept of *trivium* arose in the 8th and 9th centuries, following extended use of its elder brother, *quadrivium*. *Trivium* comprised the disciplines of grammar, logic and rhetoric, and was an initiation into the liberal arts, as well as an introduction to *quadrivium*. This was considered more complicated, and herein lies a prejudice that has been transmitted throughout the years. The term would give rise to the word 'trivial,' on account of its lesser standing when compared with *quadrivium*. The senior concept, *quadrivium* comprised the disciplines of arithmetic, geometry, astronomy and music, which completed education in the liberal arts. Boëthius systematised its use between the 5th and 6th centuries, although the concept is much older and was already present in the writings of Pythagoras.
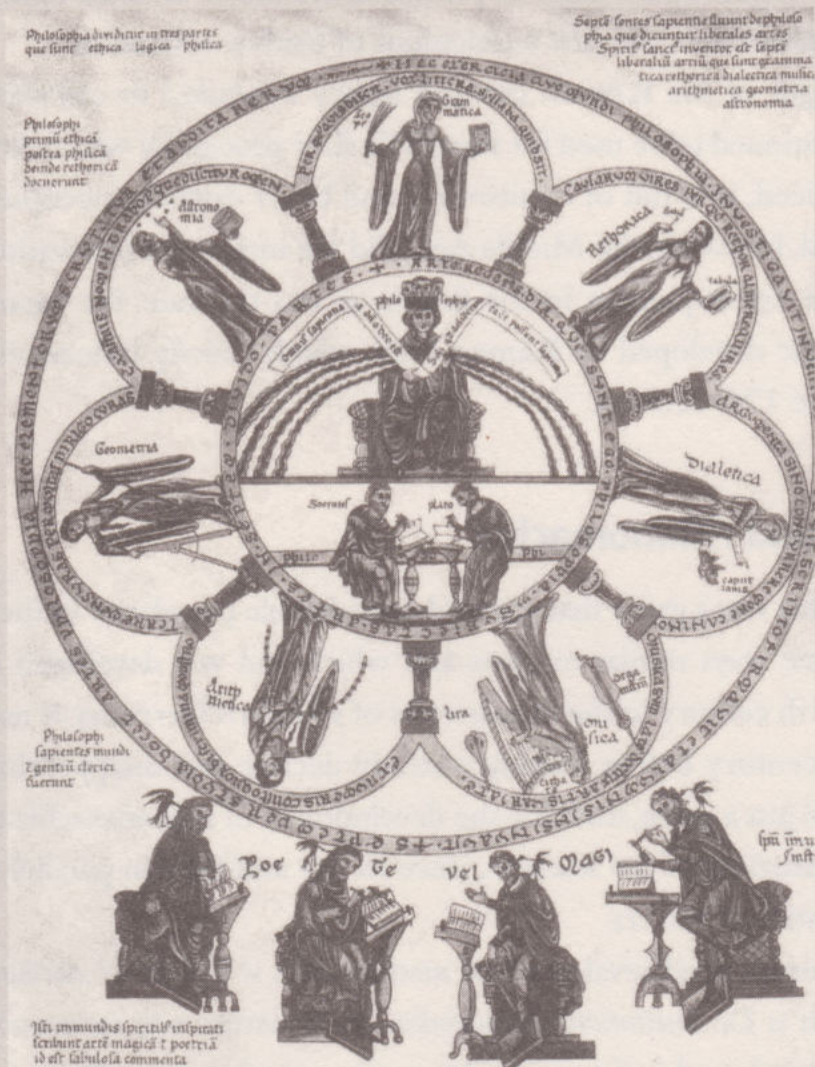


*An illustration of the seven liberal arts from the book* Hortus Deliciarum *by Herrad von Landsberg, compiled for teaching at the end of the 12th century.*

## BOËTHIUS (480–524)

Anicio Manlio Severino Boëthius, was a Christian philosopher born into a distinguished family that had produced a number of Roman emperors. His best known work is *De Consolation Philosophiae* (*The Consolation of Philosophy*), which he wrote while in prison and describes the inequalities of the world in line with Platonic concepts. He translated many Greek works into Latin in order to pass Greco-Roman culture down to future generations, since the Western Roman Empire had come to an end four years before his birth, when its emperor Romulus Augustus was overthrown by the German king Odoacer. Many of his translations were not textual and included copious personal annotations. Hence, for example, *De Institutione Arithmeticae Libri II*, which purported to be a translation of Nicomachean arithmetic, is filled with his own material. At any rate, his translations were widely used during the Middle Ages in Europe



*Boëthius in jail, a miniature from a 14th century edition of* The Consolation of Philosophy.

of proofs and propositions, which represented the great progress made by Greek thought many years before. In this respect, rithmomachia offered salvation, for the game was used as a tool for teaching students about Boëthius' concepts and ratios.

The more advanced Greek texts would be recovered with the passing of time, surpassing the mediaeval style of practising mathematics. Rithmomachia also vanished as these steps forward were taken. So completely had the game fallen out of practice, in fact, that Leibniz – despite one of his great mathematical discoveries being based on the achievements of mediaeval mathematics – was not aware of how the game worked, although he had heard of it.

In Boëthius's mathematics, numbers can be equal (*aequalis*) or unequal (*inaequalis*). Equality is not divided into categories since it is an indivisible concept, however inequality can be categorised. The major category (*maioris*) applies when one number exceeds another; the minor category (*minoris*) is used when a number is lower. These categories are subdivided into another five, depending on the type of relationship that can be established between the numbers. The major category contains the subcategories multiple (*multiplex*), superparticular (*superparticularis*), superpartient *(superpartiens)*, multiple superparticular (*multiplex superparticularis*) and multiple superpartient (*multiplex superpartiens*). The minor category contains the following subcategories: *submultiplex, subsuperparticularis, subsuperpartiens, submultiplex superparticularis, submultiplex superpartiens.*

When it comes to trying to clarify some of these concepts, a game such as Rithmomachia leaves much to be desired in understanding his system. For the late-Roman author, a multiple was, as the name suggests, when the first number was $n$ times the second, hence the relationship could be double, triple, quadruple, etc. For example, 8 is quadruple 2. A number was referred to as superparticular when the first number contained the second number and a part thereof. For example, 9 is superparticular to 6, since $6 = 6 + (1/2) \cdot 6$. A number was referred to as superpartient when the first number contained the second number and multiple parts thereof. For example, 9 is superpartient to 7, since $9 = 7 + (2/7) \cdot 7$. Multiple superparticulars contained a number multiple times and a part thereof, and multiple superpartients contained a number multiple times and various parts thereof. Hence, 15 is a multiple superparticular of 6, since $6 + 6 + (1/2) \cdot 6$, and 16 is a multiple superpartient of 7, since $7 + 7 + (2/7) \cdot 7$.

Boëthius also defined three types of mean. The first is the arithmetic mean, defined as $m = (a + b)/2$, whose main property is that it is the midpoint of two limit numbers. The second is the geometric mean, defined as $m = \sqrt{(a \cdot b)}$, the property of which is that the ratio between $a$ and $m$ is the same as the ratio between $m$ and $b$. In other words $a/m = m/b$. The third is the harmonic mean, $m = 1/((1/a + 1/b)/2)$, equivalent to $m = 2\,ab/(a + b)$.

How did rithmomachia help shed light on this tangled web of numerical relationships? Obviously by transforming them into a playful activity that deals with all these concepts. The game was played on an 8 by 16 board although the second dimension could vary. Each player had 24 counters with numbers that corresponded to the *multiplex*, *superparticularis* and *superpartiens* of established numbers. The players used various moves to capture their opponent's pieces. For example, if piece 4 was

## MEANS IN BOËTHIUS'S *ARITHMETIC*

"It is admitted and is well known among the ancients, and also features in the science of Pythagoras, Plato and Aristotle, that there are three means: arithmetic, geometric and harmonic. [...] The *arithmetic mean* refers to when, among three or any number of terms, we find an equal difference across all the terms available. [...] Now let us explain the *geometric mean*, which might better be referred to as the *proportional* mean because it takes into consideration proportions, both greater or smaller. Here, equal proportions are always considered [...], for example 1, 2, 4, 8, 16, 32, 64, or the triple proportion 1, 3, 9, 27, 81, or the quadruple, quintuple or any other multiple. [...] Together with the other means, the *harmonic* mean is that which is neither constructed with differences or equal proportions, but places the largest term with the smallest (as a quotient), and compares (or equates) the largest difference with the mean against the difference between the mean and the smallest. Consider, for example 4, 5, 6 or 2, 3, 6. Since 6 is greater than 4 by one third (i.e. 2), 4 is greater than 3 by a quarter (i.e. 1), 6 is greater than 3 by its half (i.e. 3), 3 is greater than 2 by one third (i.e. the unit)."

placed 9 positions from the piece 36, the piece 36 was captured (since $36 = 4 \cdot 9$). If the pieces for 4 and 8 were placed by 12, this was captured (since $12 = 4 + 8$). Finally, some of the conditions for finishing the game corresponded to Boëthius' three means. For example, if a player managed to position pieces $2 : 4 : 6$ in successive positions with one of their opponent's pieces between them, this meant the game was over. Why? Because 4 is the arithmetic mean of 2 and 6.
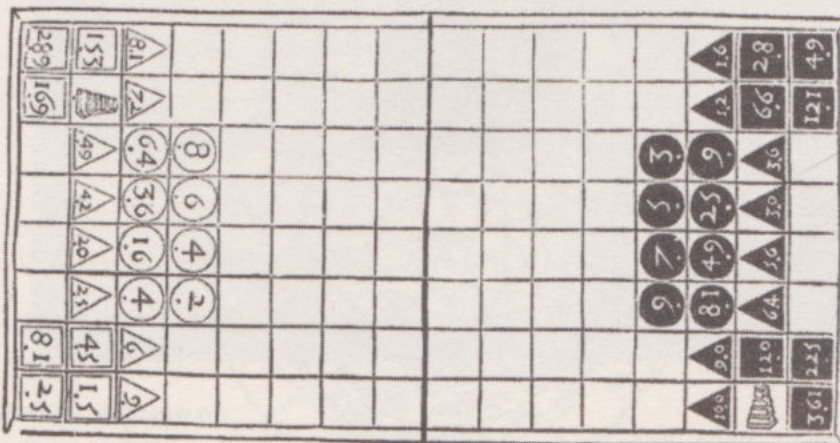


*An engraving from 1554 showing the starting positions for a game of rithmomachia.*

## BOËTHIUS UPDATED

We can use current notation to express the properties used by Boëthius to define the arithmetic, geometric and harmonic means. Consider the quantities: $a$, $b$ and $c$; let $a$ be the largest value; $b$, the middle and $c$, the smallest, hence we have the inequality $a > b > c$. We can already see that $b$ will be the arithmetic, geometric and harmonic mean of the other two quantities.

The arithmetic mean satisfies the condition that the difference between consecutive terms is the same. Hence: $a - b = b - c$. This occurs when $b = (a + c)/2$, an expression that can be deduced from the first equation above.

The geometric mean satisfies the condition that the proportions between consecutive terms are the same. Hence $a/b = b/c$. This inequality implies that $ac = bb$, hence, $b = \sqrt{(a \cdot c)}$.

The harmonic mean satisfies the condition that, in the words of Boëthius, the ratio between the larger and smaller terms is the proportion between the difference between the larger and the mean against the difference between the mean and the smaller. Expressed mathematically, this relationship is as follows: $a/c = (a - b)/(b - c)$. From this expression, we can obtain the equality $a(b - c) = c(a - b)$, from which we can also observe that $ab - ac = ca - cb$; or put another way: $ab + cb = 2ac$. Solving for $b$ in this last equation gives $b = 2ac/(a + c)$, which now gives an expression for the harmonic mean in terms of $a$ and $c$. However the expression $b = 2/(1/a + 1/c)$, obtained from the previous one by dividing the numerator and denominator by $ac$ is more common.

# Ramon Llull

*Ars Magna et Ultima* by Ramon Llull contains a system of logic for proving scientific truths. Llull's motivation was to use the work as an intellectual arsenal for debates with Muslims on the supremacy of the Christian religion. One of his innovations

*An example of the Lullian Circle from Ramon Llull's* Ars Magna.

was the Llullian Circle, which was constructed of discs showing symbols relating figurative concepts that were so arranged that they always created valid combinations – at least valid according to Llull – when the discs were rotated.

The innovation of Llull's logic lay in its orientation towards studying the properties of concepts. Therefore it could be understood as a synthetic logic at a point in history that was intellectually dominated by analytical logic. This different perspective attracted the interest of thinkers such as Giordano Bruno (1548–1600) and Gottfried Wilhelm Leibniz (1646–1716), who returned to Llull's ideas in the context of philosophy. Leibniz made use of them in his famous work *De Arte Combinatoria*, published in 1666. In fact, this line of thought has survived to the present day, and it is not for nothing that it forms the basis of the logic systems of many current computing tools.

*A miniature taken from the* Ars Magna *by Ramon Llull.*

## ANALYTIC AND SYNTHETIC LOGIC

The philosopher Immanuel Kant (1724–1804) described the distinction between synthetic and analytical logic in his *Critique of Pure Reason*. The distinction is based on the dichotomy of whether the concept of the predicate is contained in the concept of the subject. A proposition is said to be analytic when there is such containment, and synthetic otherwise. For example, the proposition 'all triangles have three sides' is analytic because the fact of having three sides is contained in the definition of a triangle. When this is not the case, the proposition is synthetic. For example, 'certain teachers fail a lot'. However, in 1951, the North American Willard van Orman Quine (1908–2000), analytic philosopher and teacher of Noam Chomsky, dared to dispute this distinction.

Ramon Llull's prescience with respect to modern science is not just limited to his ideas about logic. He also developed an electoral system for making church appointments. The system for electing an abbess is described in his *Blanquerna* and is developed in a more academic formulation in his works *Ars Electionis* and *Artifitium Electionis Personarum*. Llull's ideas in this respect exerted a great influence on Cardinal Nicholas of Cusa (1401–1464), a philosopher and theologian regarded as the father of German philosophy. In fact, the election systems proposed by both of these figures would eventually give rise to modern proportional representation systems. Llull's electoral system satisfied the criteria of Condorcet's system (1785), and Cusa's is related to Borda's counting method (1770). All these systems sought to determine the element most preferred by a group of people based on the multiple preferences of each elector.

## The introduction of Arabic numerals

From its distant origins in India, the system of numerals we use today reached Europe through Islamic scholars in North Africa. This explains why the numbers are often referred to as 'Arabic numerals'. The system was designed and developed by the learned Persian Muhammad ibn Musa al-Khwarizmi.

Gerbert d'Aurillac, who adopted the name Sylvester II when he was appointed as Pope, played a fundamental role in the process, favouring the reintroduction of the abacus in Europe and promoting Arabic numerals. Aurillac's abacus was a renewed instrument, which, in contrast to Roman abacuses, was based on nine numerical symbols and represented zero using an empty column. Its use became widespread

in Europe in the 11th century, although the abacus with Arabic numerals did not replace its Roman numeral counterpart, since it was believed that while the new abacus was a superior calculation tool, Roman numerals represented the only possible way of writing the results.

However, the main responsibility for the spread of Arabic numerals lies, without a doubt, with Muhammad ibn Musa al-Khwarizmi. His main work is *Hisāb al-ŷabr wa'l muqābala (The Compendium on Calculation by Restoration and Balancing)*. It is an early and much more important text than the *Principles of Hindu Reckoning*, written by Kūshyār ibn Labbān, of which unfortunately no Arabic version has survived, only the later Latin translations from the 12th and 13th century. The scale of his contribution to the history of mathematics is clear from the title: *al-ŷabr* would give rise to the word 'algebra', and the author's name to the term 'algorithm'.

## GERBERT D'AURILLAC (946–1003)

The future Sylvester II left the monastery of Saint Gerald de Aurillac to follow the Count of Barcelona, Borrell II, to the monastery Santa María de Ripoll, where he studied mathematics for three years. During this period, he travelled to Cordoba and Seville, where he learnt mathematics and astronomy from Arab teachers, but above all he became convinced of the excellence of the numeral system they used. Gerbert d'Aurillac wrote a number of works on mathematics and astronomy, largely dealing with the *quadrivium*, or rather works written for students, not scholars. His works can be found in volume 139 of *Patrologia Latina*, a compilation of papal writings from the times of Tertullian (160–220) to Innocent III (1160–1216). As well as reintroducing the abacus, Gerbert d'Aurillac also reinstated the armillary sphere as a learning aid for his students.

*Statue dedicated to Pope Sylvester II in the French municipality of Aurillac.*

## MUHAMMAD IBN MUSA AL-KHWARIZMI (780–850)

Little is known for sure about the life of Muhammad ibn Musa al-Khwarizmi. Even his birthplace, often said to be modern-day Uzbekistan, is open to discussion. A mathematician, astronomer and geographer, al-Khwarizmi is regarded as the father of algebra and is credited with having introduced our numeral system. He was educated and worked in the House of Wisdom in Baghdad, a research and translation institution that has been compared to the Library of Alexandria. There he compiled and translated the great Greek and Indian scientific and philosophical works into Arabic. The institute also had an advanced astronomical observatory. Al-Khwarizmi wrote a vast number of works, many of which are of fundamental importance. He also wrote a political history. He is regarded as one of the great sages of all time.

*A 1983 Soviet stamp honours Muhammad ibn Musa al-Khwarizmi at a time when Uzbekistan was part of the USSR.*

In addition to his work on algebra, al-Khwarizmi wrote a work on arithmetic, entitled *Kitab al-Ŷamaa wa al-Tafriq bi Hisab al-Hind* (*Book of Addition and Subtraction According to the Hindu Calculation*), in which he gives an in-depth description of the Indian positional base 10 numbering system and the methods for carrying out the main arithmetic operations. It is possible that al-Khwarizmi was the first to use zero as a positional indicator, which he shows in his work on arithmetic. Latin translations of this work spread throughout Europe and were widely used over the centuries in universities under the Latinised title *Algoritmi de Numero Indorum*.

## The rise of Arabic numerals

The introduction of Arabic numerals in Europe was neither quick nor easy, and the process was certainly not without controversy. The city of Florence prohibited their use, arguing that they made it too easy to falsify a balance. There were confrontations between abacists and algorists for a number of centuries. Ultimately, it was the latter who would win, although the dispute would not be settled until the second half of the 16th century.

Proponents of the abacus defended the use of Roman numerals, which were more practical for use with that counting device. On the other hand, the algorists argued for the use of Arabic numerals, which were harder to use with the old abacus but more useful on paper. History has termed them the algorists, since the operation on paper is algorithmic, in the sense that the operations follow an algorithm. Both groups distributed their own writings on how to use the abacus, in the case of the former, and how to operate using pencil and paper (or similar media, such as parchment or slate) in the case of the latter. The texts of the abacists attached little importance to zero and gave priority to multiplication and division above other operations; they also focused on duodecimal fractions (base 12). The algorists, on the other hand, extolled the virtues of the usefulness of zero, considering and favouring many more operations (addition, subtraction, multiplication, division, division and multiplication by two, and finding roots) and focused on sexagesimal fractions.

In the end, as always, the matter was settled by money. In Italy, the balance began to tip in favour of the algorists, as it slowly became clear to merchants that Arabic numerals were much easier and faster to use. The Italian enthusiasm for Arabic numbering began to take hold in the rest of Europe. The new methods for operating were introduced in Germany in 1200, reached France towards 1275, and arrived on the shores of Britain in 1300.

The epicentre of this Italian mathematical earthquake that dispersed Arabic numerals was a man: Leonardo of Pisa, better known as Fibonacci. Fibonacci's *Liber Abaci* uses Arabic numerals to describe the commercial applications of arithmetic. It sets out the algorithms for operations with the explicit purpose, as stated in the preface, that they would be used in Italy on account of their usefulness. The *Liber Abaci* was the first book written in Europe to make use of Arabic numerals.

The *Liber Abaci* laid the foundations for a new genre of mathematical works that rose to the fore between the 14th century and the second half of the 16th. They focused on commercial arithmetic, bookkeeping techniques that promised more profits for the money minded. The proliferation of this type of arithmetic is associated with the expansion of abacus schools, above all in Italy. In 1340, there were six abacus schools in Florence with 1,200 students (a considerable number bearing in mind the city had a population of 100,000 inhabitants). The schools, such as Galigai in Florence, of which considerable records have been preserved, taught basic arithmetic to students aged ten or eleven for a period of two or three years. The students normally came from a grammar school where they had started to read and write at the age of five or seven. After abacus school, at the age of thirteen or

## LEONARDO OF PISA (1170–1250)

Leonardo of Pisa, better known as *Fibonacci*, was the son of Guglielmo Bonacci, an Italian merchant based in Bugia. His name is derived from *figlio di Bonacci* (son of Bonacci). Leonardo learnt the Arabic numerals with his father. Later, anxious to expand his knowledge, he travelled to Egypt, Syria and Byzantium, where he immersed himself in the study of Arab mathematics. His work collects together and describes the wealth of knowledge he acquired on his journeys. In addition to the *Liber Abaci*, he wrote *Liber Quadratorum* (1225), which dealt with algebra, *Practica Geometriae* (1223) and many other texts.

fourteen, the students began apprenticeships as craftsmen, bookkeepers or money lenders. A select few were spared from entry into the workplace and went on to study the classics.
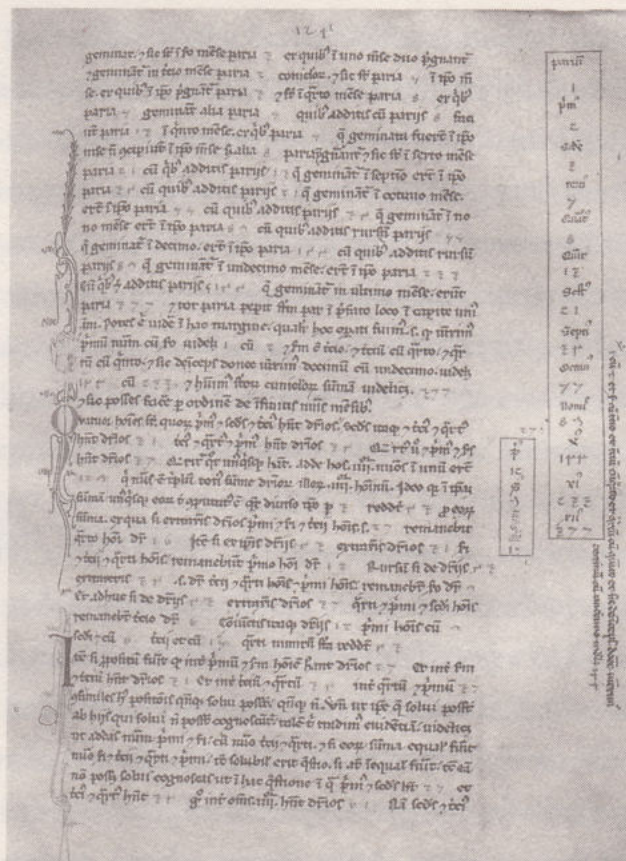
*Page from Fibonacci's* Liber Abaci.

Commercial arithmetic and the abacus schools influenced mathematical development at the time. It was often the case that the teachers from the schools were called in to solve practical problems. This was the case with Giovanni di Bartolo, a teacher at an abacus school in Florence, whose calculations contributed to the construction of the dome of the city's cathedral in 1420. However, this practical mathematical activity took place independent of the academic activity at universities. In fact, there was practically no relationship between the teachers at abacus schools and academic staff from the universities. The majority of universities continued the classical arithmetic tradition of Boëthius and the Roman numerals.

The spread of Arabic numerals also appeared in Italian manuals referred to as *Pratice della mercatura*. The best known are *Libro di divisamenti di apesi e di misure di mercantie*, written by Francesco Balducci Pegolotti, and published in the first half of the 14th century, as well as works by Antonio da Uzzano (1442) and Giorgio di Lorenzo Chiarini (1458).

Commercial arithmetic was extremely popular, but on account of the high cost of textbooks, it remained beyond the reach of students, although if they were enrolled in an abacus school it is likely that copies were used for teaching. It is believed that reference books were kept by merchants and business premises, where, to a certain extent, they acted to uphold the legitimacy of the profession.

## MATHEMATICS IN THE TRANSITION PERIOD

The era following the publication of the *Liber Abaci* is regarded as a transitional phase between one system and the other, a change of model. Scholars have attempted to classify and give order to the proliferation of books and treatises published at that time. In doing so, they have identified four different types of work:

- Theoretical treatises that followed the writings of Boëthius.
- Arithmetic abacus writings that described abacus-based computation.
- Algorithms that used Arabic numbering and described how to carry out calculations on paper. These were based on the work of al-Khwarizmi.
- Computing texts that described the calculation systems for determining the ecclesiastical calendar.

# THE PRINTING PRESS

The invention of the movable-type printing press is attributed to Johannes Gutenberg (1398–1468), who around 1450 developed the machine in the German city of Mainz. The first texts were printed between 1449 and 1450, when the *Constance Missal* – regarded as the first book to be printed using movable type – was published, and 1454 or 1455, when he completed printing the famous 42-line Bible (so-called because of the number of lines on each page). It had 1,282 pages in total, split into different volumes (normally two). There are currently 48 copies of the Gutenberg Bible. Their price at the time of printing was equivalent to three times the annual salary of an average worker. In spite of the impossibility of assessing the importance of the movable type press – which allowed the mass production of texts and as such represents one of humankind's greatest cultural revolutions – Johannes Gutenberg died in poverty.
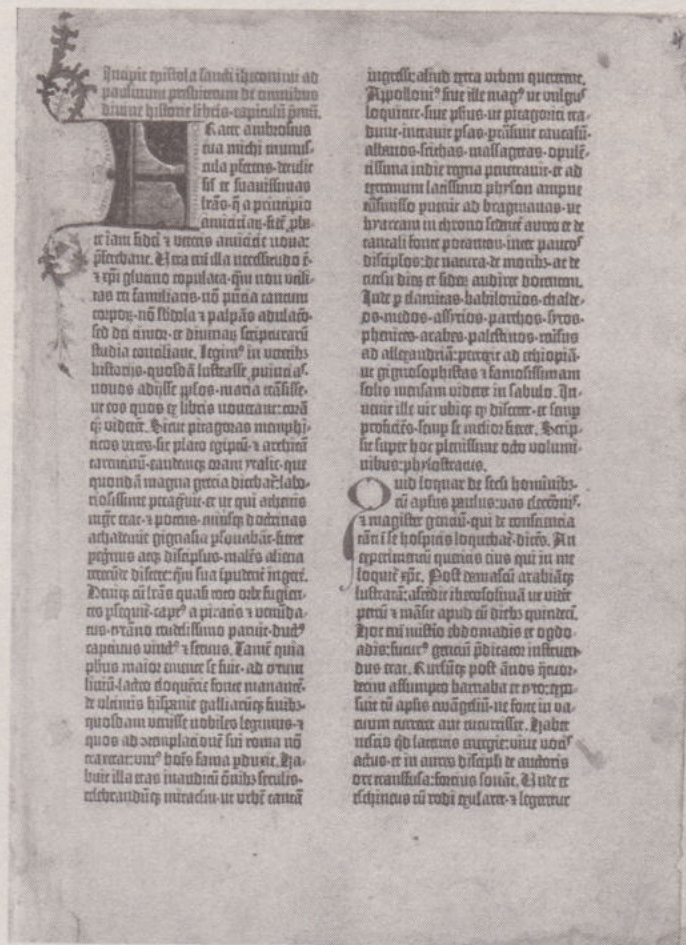


*A page from a Gutenberg Bible.*

The first book of mathematics to be printed was *The Treviso Arithmetic*, named after the city where it was published in 1478. Second on the list is *Summa de l'art d'aritmètica*, written by Francesc Santcliment and published in Catalan in Barcelona in 1482. *Rechenbuch* by Ulrich Wagner, published in Bamberg (Bavaria) in 1483, was the third to be printed.



*An engraving from* Margarita Philosophica *(1508) by Gregor Reisch, which shows Boëthius and Pythagoras in the middle of a calculating competition supervised by the goddess of arithmetic. Note that Boëthius, on the left, is calculating using Arabic numerals, while Pythagoras is making use of an abacus.*

A testimony to the importance attributed to commercial arithmetic over pure mathematics texts at that point is the fact that Euclid's work, *Elements*, was not printed until 1482, appearing in a Latin translation under the title *Elementa Geometriae*. Boëthius' *Aritmetica* was not printed until 1488. The first printed algebra text was *Summa de Arithmetica, Geometrica, Geometria, Proportioni et Proportionalità* by Luca Pacioli, which was published in Venice in 1494 and throughout the

16th century. It generated a large series of explanatory texts because the level of Pacioli's work was considerably high. However, despite the importance of these additions, at that time the majority of books that were published were related to commercial arithmetic.



*A portrait of the mathematician Luca Pacioli painted by Jacopo de'Barbari around 1496.*

## THE BUSINESS OF MATHEMATICS

Manuscript 102 (A. III 27) held at the library of the Accademia degli Intronati in Siena, Italy, is one of the four manuscripts dealing with commercial mathematics prior to the year 1500 that has survived to the present day. It includes the following problem, which we show here: 'If you wish to know how much money someone has in their bag, do the following. Suppose they have 4, tell them to double it, to give 8, and then add 5, to give 13, then multiply everything by 5, to give 65, add 10, to give 75, then multiply this by 10, to give 750, now subtract 350, to give 400, which corresponds to 4, and as each hundred corresponds to a number, 400 gives 4'.

## Fractions and decimals

When Arabic numerals reached the West they only represented whole numbers. Fractions continued to be represented using sexagesimal notation, as in Babylon. Kūshyār ibn Labbān describes the procedure in his book *Principles of Hindu Reckoning*, showing numbers as degrees: 1/60 as minutes (*daqā'iq*), 1/(60²) as seconds (*thawānī*), 1/(60³) as thirds (*thawālith*), 1/(60⁴) as fourths (*rawābi'*), etc. The measurements were already indicated using symbols that are highly familiar today: degrees were indicated using °, minutes with ', seconds with ", thirds with ''', etc.

It was not until the 16th century that Simon Stevin wrote a work that reinforced the importance of decimal notation for fractions, and began to campaign for governments to adopt the system. Prior to Stevin, decimal notation had already been used for fractions although it was not widespread. The Persian mathematician and astronomer Ghiyath al-Kashi (1380–1429), from the great academy of Samarkand, had already used it a century earlier in his works on trigonometry and for calculating the number π. Al-Kashi was also aware of what is now referred to as Pascal's or Tartaglia's triangle.

### SIMON STEVIN

The Flemish mathematician, engineer, physicist and even semiologist, Simon Stevin (1548–1620) published *De Thiende* (*The Tenth*) in 1585, a work that dealt with decimal notation and how to use it to carry out calculations. He was the first mathematician to recognise the validity of negative numbers, accepting them as the result of the problems with which he was working. He also developed the algorithm for obtaining the greatest common divisor of two polynomials. He wrote all his works in Dutch with the intention that they could be understood by craftspeople. As a result his texts were extremely popular and sold quickly, and contributed to the success of decimal notation.

## The number $\pi$

As has already been mentioned, the Persian, al-Kashi, also contributed to calculating the decimal points of $\pi$. While Si Zu Chongzhi calculated them using the regular polygon with $12{,}288 = 3 \cdot 2^{12}$ sides, al-Kashi did so using a polygon of $805{,}306{,}368 = 3 \cdot 2^{28}$ sides, allowing him to obtain the number correct to 14 decimal points, around the year 1430.
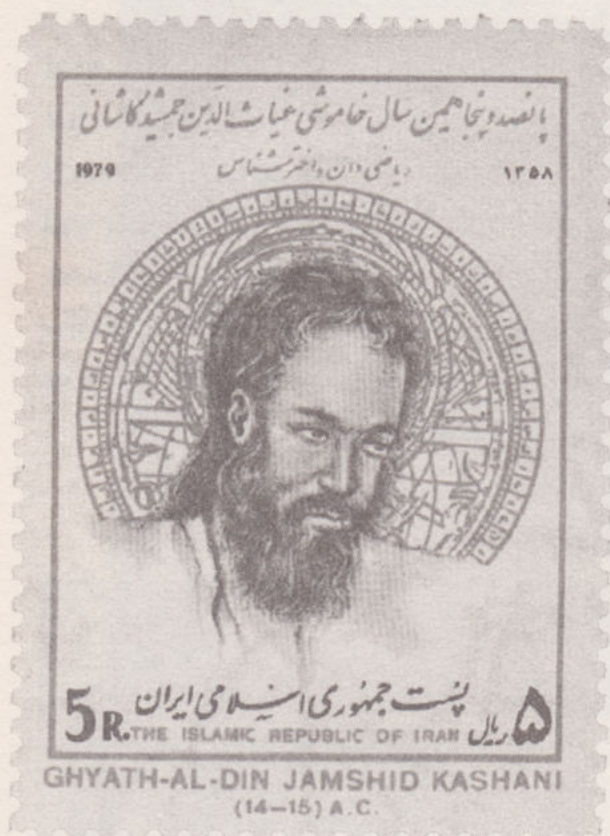


*The mathematicians al-Kashi and Ludolph van Ceulen increased the number of decimal points known for $\pi$.*

From his position at the University of Leiden in the Netherlands, Ludolph van Ceulen continued with al-Kashi's calculations, and in 1596 obtained the number correct to 20 decimal places, with a polygon of $515{,}396{,}075{,}520 = 60 \cdot 2^{33}$ sides. Later, in 1615, he obtained 35 decimal points with a polygon of $4{,}611{,}686{,}018{,}427{,}387{,}904 = 2^{62}$ sides.

The system for approximating the number $\pi$ using polygons gave good results correct to many decimal places. However many mathematicians believed it was possible to discover more efficient alternatives. They considered the possibility of finding $\pi$ by adding or multiplying a series of infinite terms. The first to discover an

## FRANÇOIS VIÈTE (1540–1603)

Viète was a lawyer and held a position in the Spanish parliament. However, above all he was a renowned mathematician and a pioneer in representing the parameters of an equation using letters. He distinguished himself as a codebreaker, deciphering secret codes using statistical techniques. He was able to decipher the highly complex Spanish code to give the French the advantage in a war against Spain. Shortly before his death, he wrote a memo on cryptography, which rendered the methods used for coding at the time obsolete.

expression of this type in Europe was François Viète. He was one of the fathers of modern algebra, albeit one who nonetheless was unaware of the expression derived by Madhava of Sangamagrama and described in the previous chapter. Viète's method was based on a product of infinite terms that was in turn based on the square root of two. While this did not make it easy to find the decimals of $\pi$ the definition opened up new paths for approximating the elusive number to a greater number of decimal places. At any rate, it was the first infinite product to give an expression for $\pi$ in the history of mathematics. The expression was as follows:

$$\pi = 2 \cdot \frac{2}{\sqrt{2}} \cdot \frac{2}{\sqrt{2+\sqrt{2}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}} \cdot \frac{2}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{}}}}} \cdots$$

# Chapter 3
# The First Mechanical Calculating Instruments

The introduction of Arabic numerals resulted in improvements to calculation techniques and produced a knock-on effect on the field of science in general. In the 17th century, many scientific advances combined to alter our view of the universe. The period is often referred to as the Scientific Revolution and it paved the way for the Enlightenment of the 18th century. Human knowledge was evolving extremely quickly at this time. Science generated new calculations and drove the development of increasingly powerful, sophisticated and precise instruments for doing so. Since manual calculations are always subject to human errors, the goal of reducing the involvement of people to a minimum spurred on the mechanisation of calculating instruments. The first mechanical calculating instruments were built during the 300 year period spanning the 17th, 18th and 19th centuries.

## The 17th century

In 1617, the Scottish mathematician John Napier invented one of the first calculating tools, an abacus known as 'Napier's Bones'. The instrument was so efficient that it continued to be used for some applications until the start of the 20th century.

### JOHN NAPIER (1550–1617)

The mathematician John Napier discovered the theory of logarithms, which he called 'artificial numbers,' and gave his name to the Napierian logarithms. With a strong interest in theology, Napier applied mathematical formalism to an interpretation of the Apocalypse of Saint John, that led him to calculate that the end of the world would occur between 1688 and 1700.

Napier's Bones were based on a concept that was actually very simple. They were a sort of multiplication table. The instrument had nine wooden bars that formed a square, numbered 1 to 9. Each row had spaces to display the nine multiples of numbers across the top, represented where necessary as two digits separated by a slanted line, as shown here.
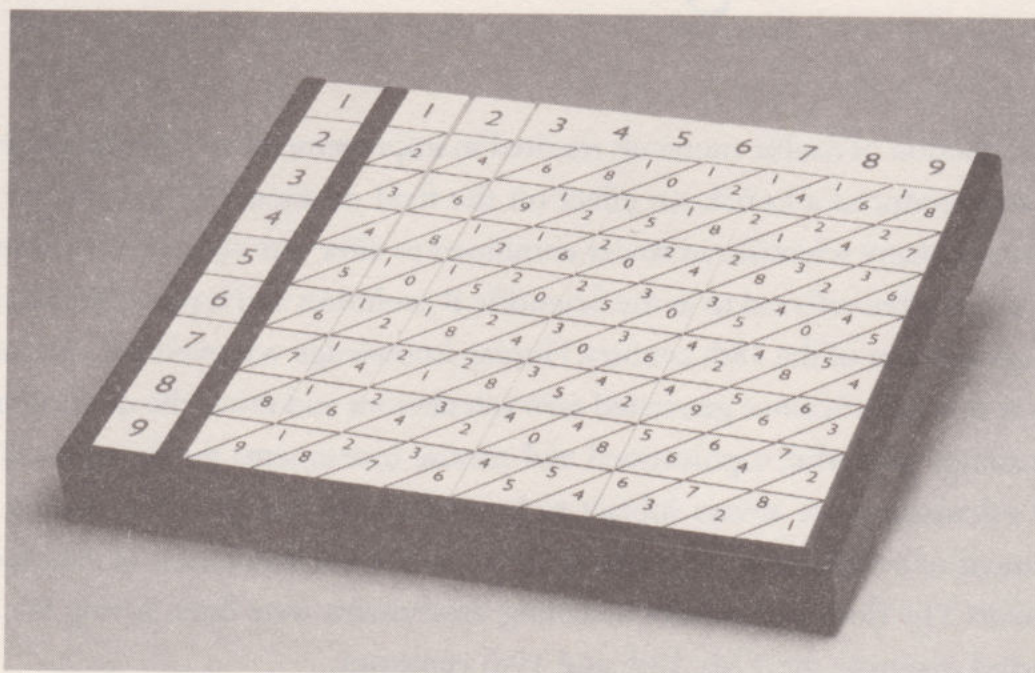


*A modern replica of Napier's Bones*

To see the instrument in operation, let us consider the multiplication of 35,672, a number that puts all the bars (or columns) to use. The bars are arranged in the order of the five digits in the number, i.e. the first bar should have the number 3, followed by 5, then 6, then 7, and finally 2. By studying the bars in this layout, we can see all the possibilities for multiplying 35,672 by any number from 1 to 9, which appear in each row beneath. To multiply 35,672 by 4, we take the numbers that appear in row number four, which in this case correspond to:

$$1/2 \ \ 2/0 \ \ 2/4 \ \ 2/8 \ \ 0/8.$$

Now we add the numbers that neighbour each other within those diagonal divisions:

$$1/2 + 2/0 + 2/4 + 2/8 + 0/8.$$

This gives:

$$1 / 4 / 2 / 6 / 8 / 8.$$

Hence, the answer is 142,688. Check it with a calculator – or by hand – if you wish.

$$35,672 \cdot 4 \ = 142,688.$$



*Napier's Bones used to multiply 35,672 by 1 to 9.*

When it comes to multiplying by numbers with more than one digit, the operation is essentially the same. Each digit of the multiplying number is used to multiply by the initial target number and the results are then added together. All the partial multiplications required can be found in the bars. To multiply 35,672 by 436, we carry out the described process for the rows 4, 3 and 6 on the bars. This will give the following numbers, with the inclined divisions aligned in accordance with the positions of the corresponding digits in the multiplying factor (436).

$$1/2 \ 2/0 \ 2/4 \ 2/8 \ 0/8$$
$$0/9 \ 1/5 \ 1/8 \ 2/1 \ 0/6$$
$$1/8 \ 3/0 \ 3/6 \ 4/2 \ 1/2$$

Having arranged the numbers in this way, it is now possible to calculate the multiplication of 35,672 by 436 as the sum of the results of the partial multiplications within each divided area. This is shown below, first giving the results of the partial multiplications, then the partial sums, and the result after values have been carried.

$$1/2 \quad 2/0 \quad 2/4 \quad 2/8 \quad 0/8$$
$$0/9 \quad 1/5 \quad 1/8 \quad 2/1 \quad 0/6$$
$$1/8 \quad 3/0 \quad 3/6 \quad 4/2 \quad 1/2$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$1 \,/\, 4 \,/\, 13 \,/\, 23 \,/\, 21 \,/\, 19 \,/\, 9 \,/\, 2$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$1 \,/\, 5 \,/\, 5 \,/\, 5 \,/\, 2 \,/\, 9 \,/\, 9 \,/\, 2$$

We can use a calculator to check it again. The result is correct:

$$35,672 \cdot 436 = 15,552,992.$$

It is clear that the calculated rows correspond to those that would be obtained using the modern algorithm for multiplication. The partial results are:

$$
\begin{array}{r}
35672 \\
\times \quad 436 \\
\hline
214032 \\
1070160 \\
14268800 \\
\hline
15552992.
\end{array}
$$

However, Napier's Bones were not just useful as an aid for multiplication. A number with several digits could be divided by another using the bar corresponding to the numbers in the divisor. Hence, the multiples of the divisor appear in the different lines of the bars and can be used to determine the digits in the results.

John Napier is also the father of another fundamental discovery in the story of calculation: logarithms. The Scot discovered that logarithms could be used to transform the more complicated operations into the simpler ones. Multiplication was

converted into addition, division into subtraction, exponentiation into multiplication, and roots into division. This simplification was a significant advance for carrying out complex operations by hand, and gave a deserved boost to the field of mathematics.

$$\log(a \cdot b) = \log(a) + \log(b)$$
$$\log(a/b) = \log(a) - \log(b)$$
$$\log(a^b) = b \cdot \log(a).$$

Hence, to calculate $a \cdot b$ all we need to do is calculate $e^{\log(a) + \log(b)}$.

Logarithms formed the basis of the slide rule, another great calculation instrument developed by the British mathematician William Oughtred (1574–1660), who also introduced the symbol $\times$ for multiplication and *sin* and *cos* for the sine and cosine functions. The mathematician's work was based on an earlier tool by Edmund Gunter, an instrument that used a single logarithmic scale, whereas the slide rule combined two of them. Later, in 1859, the Frenchman Amédée Mannheim made the modifications that would give it its modern form.



Portrait of William Oughtred, regarded as the inventor of the slide rule.

The slide rule was not only used for addition and subtraction, but was more suitable for carrying out multiplication and division. The most modern were even capable of operations such as roots, trigonometric functions, exponentials and

logarithms. It must be noted, however, that the precision of this system was limited. It was common to work to three significant figures, although more precise and larger slide rules could achieve a greater precision. The operator also needed to take magnitudes into account, since these were ignored when using the slide rule. The device was the chief scientific calculation tool until the development of electronic pocket calculators rendered it obsolete in the 1970s.
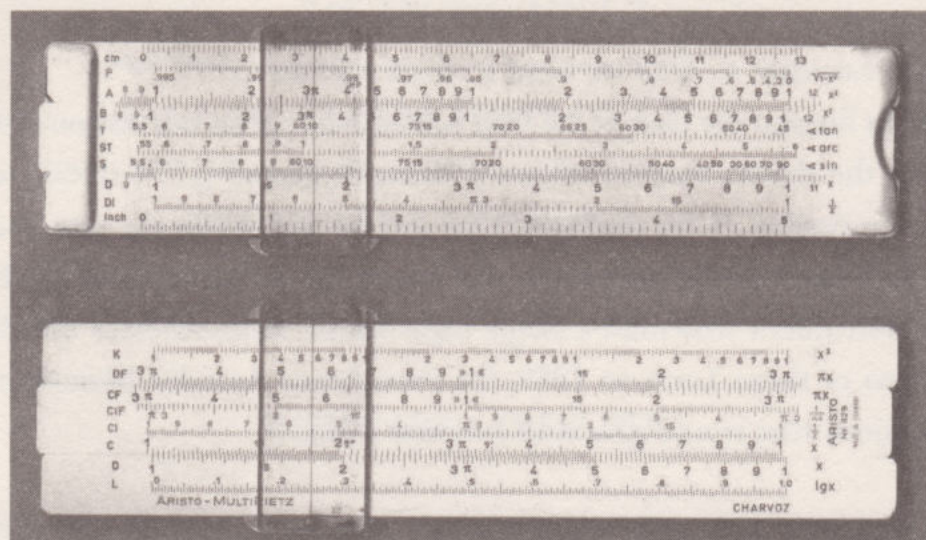


*A slide rule model from the 1960s, shortly before the device was replaced by calculators.*

## The first calculators

The first electronic pocket calculator was launched in 1972, the famous Hewlett-Packard HP-35. So far, this book has been retelling the revolution of calculation and its automation. Or more precisely it is the development of the theoretical foundations that led to the astonishing result of a small calculator that fitted into the palm of a hand, heralding the dawn of computing and the communications revolution that followed.

However, we do not need to travel as far forward as the 20th century to find the green shoot that sprouted from all the preceding theoretical seeds. Indeed, what is regarded as the first calculator in history was developed as long ago as the 17th century in the era of mechanical rather than electronic instruments. It was called the 'calculating clock' and was designed by Wilhelm Schickard (1592–1635) in 1623, in Tubinga.
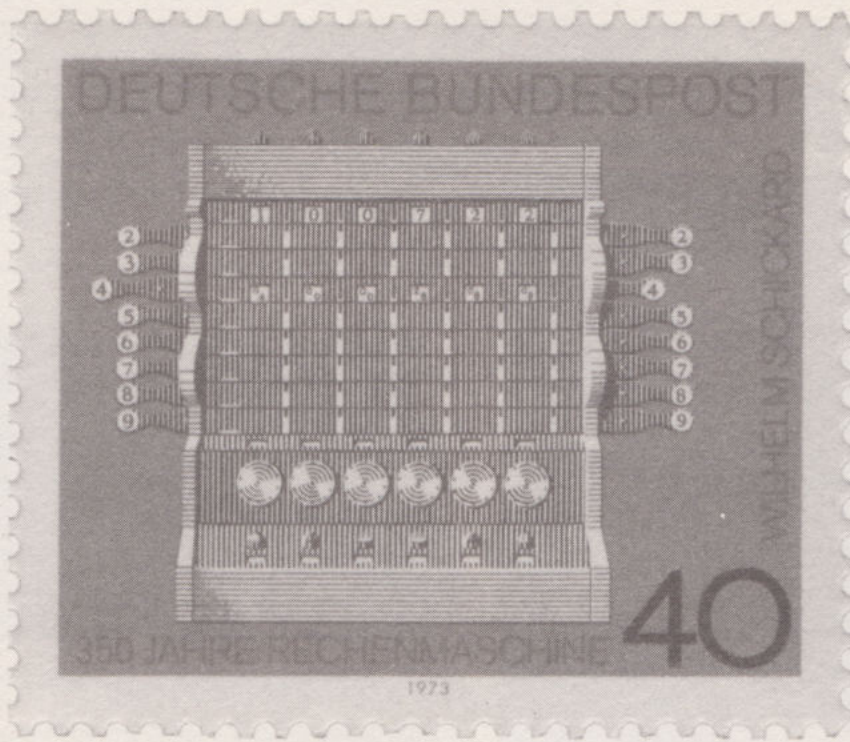
*A German stamp commemorates Wilhelm Schickard's 'calculating clock.'*

The world's first calculator made it possible to carry out the four basic arithmetic operations. Addition and subtraction could be carried out in a fully mechanical manner, but this was not so for multiplication and division, which required a human operator to carry out intermediary steps. The machine operated using elements similar to Napier's Bones, and carried values using a system of pinwheels that turned, for example, the tens counter, after a full rotation of the unit wheel. This procedure for carrying by wheels had been used in Europe since at least the 16th century to construct pedometers that counted the number of steps taken. This figure was used to infer distances travelled. The oldest known pedometer was invented by the Frenchman Jean Fernel in 1525.

Schickard's calculator did not have a huge impact on the history of computation, since its inventor apparently died as a victim of the terrible plagues that ravaged Europe at the time, and the machine slipped into obscurity until the 20th century. Nonetheless, we are aware of it thanks to the correspondence between its inventor and Johannes Kepler, with whom he worked. The letters included a number of sketches of the device, which made it possible to reconstruct it and check that it really worked. In one letter, Kepler confirms that he had requested a copy of the calculator from his friend perhaps to help with his astronomical work.

Pascal's calculator, invented by Blaise Pascal, was the first to make it big in history. The philosopher and mathematical genius presented it in public in 1642 when he was just 19 years old. The mechanism was similar to that used by Wilhelm Schickard; it also operated by incrementing a higher unit after a full turn of the previous unit. Unfortunately, this system of mechanical operation soon ran into problems caused by the poorly machined cogs failing to mesh properly.
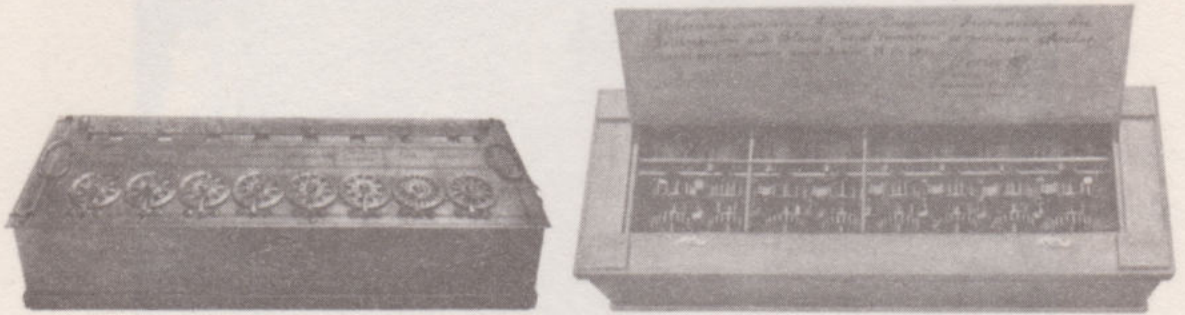


*Pascal's calculator, or Pascaline, invented by Blaise Pascal.*

Pascal developed his invention entirely independently of the work of Wilhelm Schickard. In fact, Pascal's calculator was simpler and only permitted addition and subtraction, using a method of complements for subtraction. The first version was able to operate with five digits, in contrast to the six digits of Schickard's machine, although subsequent versions allowed for operation with a greater number of digits. Although the calculators were made available for sale, their high price kept them from being a commercial success for the Pascal family. Pascal's calculator was reduced to a toy and status symbol for the wealthy classes in France and further afield in Europe. Pascal continued making improvements to his design for a decade, creating up to fifty different versions.

In spite of their faults and limitations, these devices represented a milestone. Their appearance sparked a frenzy of inventions in Europe, inspiring mathematicians and engineers to design better mechanical calculators. Some of these entailed technological solutions that represented advances on Pascal's calculator, whereas others proposed simple variations.

The Englishman Samuel Morland (1625–1695), for example, constructed a calculation machine adapted to the English currency system, which was not decimal at the time and used pence, shillings and pounds all counted in different bases. In contrast to Pascal's machine, his device did not implement automatic carrying. It had a carry wheel for each unit, and the user had to manipulate the carried values once

## BLAISE PASCAL (1623–1662)

The French mathematician, physicist, philosopher and theologian Blaise Pascal is regarded as the father of computers, together with Charles Babbage. He was a child prodigy: At the age of just 11 he wrote a short treatise on the sounds of vibrating bodies and an independent proof that the angles of a triangle added up to 180°. At the age of 12, he studied Euclid and attended meetings with the greatest mathematicians and scientists in Europe, including Roberval, Desargues and Descartes himself. He wrote his main treatises on projective geometry at the age of just 16. When Descartes read the manuscript, he refused to believe it could be the work of a teenager. Pascal was a mathematician and physicist of the first order, whose achievements shone brightly amongst the science of the time.
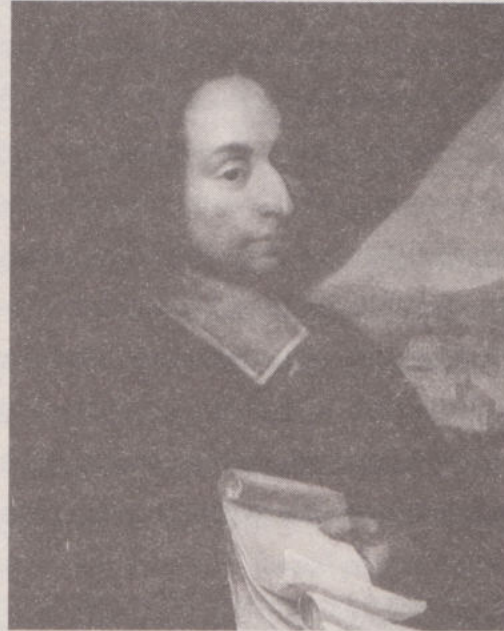
the partial additions were complete. However, Morland's most spectacular achievement was to build a calculator small enough to fit in one's pocket.
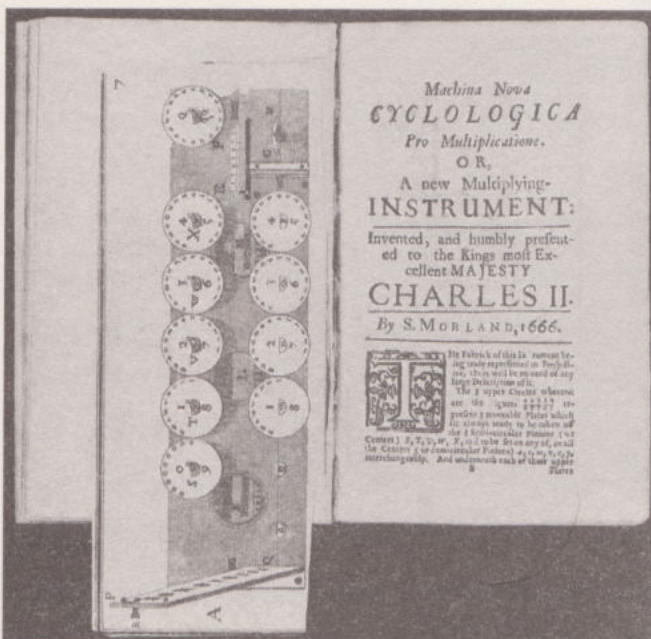
*The book* The Description and Use of two Arithmetick Instruments, *published in London in 1673, described the calculating machine invented by Samuel Morland.*

A calculator developed by Gottfried Leibniz represented a considerable advance on Pascal's machine, since it allowed multiplication to be carried out automatically. Until that point, multiplication using a calculator was tedious, requiring a human operator to make intermediate steps. However, the problem was the same as always – technical complications prevented the machines from working properly. The precision of the parts was not sufficient to ensure the machine had the required reliability. Even still, the improvements introduced by Leibniz had a great impact on subsequent developments. In the main this was the innovation that permitted automatic multiplication – Leibniz's cog assembly, which used multiple cogs with teeth placed at increasing distances. The improvements in manufacturing required to ensure this component was reliable were not consolidated until 1822, when the Frenchman Charles-Xavier Thomas de Colmar invented and later commercialised the Arithmometer.

However, Leibniz's contributions to calculation went far beyond building a somewhat unreliable calculating machine. Indeed, much more important, for example, was his work on the binary numbering system, upon which modern computing is based. This system of numbering had been studied by the Englishman
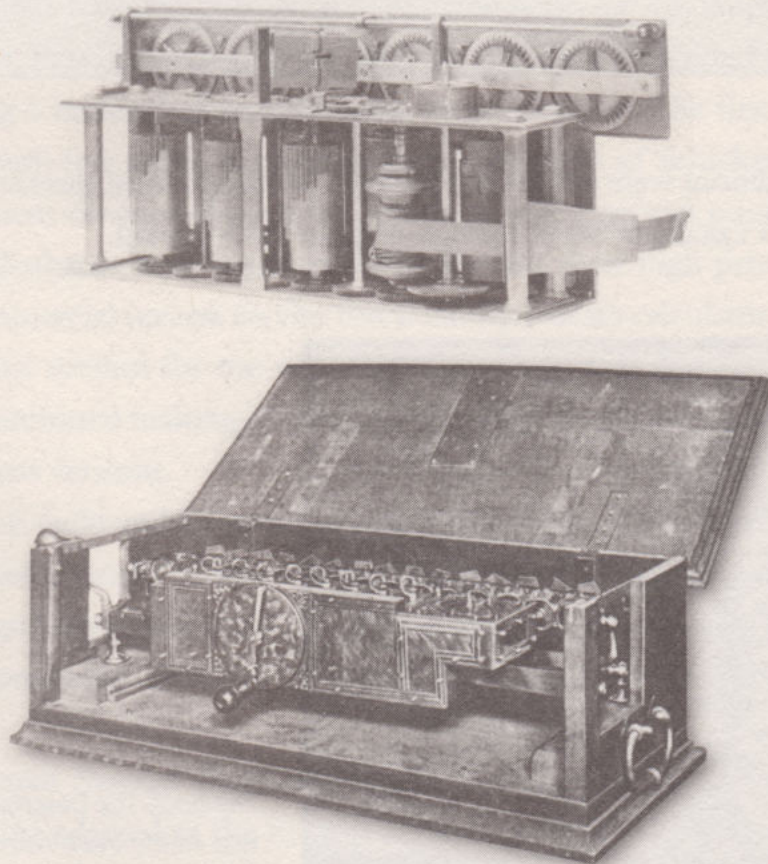
*The inner workings of Charles-Xavier Thomas de Colmar's Arithmometer (above), and the calculator invented by Gottfried Leibniz.*

Thomas Harriot (1560–1621), but his ideas went unpublished. The binary system only uses two digits, a zero and a one, which are used to represent numbers. The following table shows the binary representation of the numbers from 1 to 16.

| 0 = 0 decimal | 100 = 4 decimal | 1000 = 8 decimal | 1100 = 12 decimal |
|---|---|---|---|
| 1 = 1 decimal | 101 = 5 decimal | 1001 = 9 decimal | 1101 = 13 decimal |
| 10 = 2 decimal | 110 = 6 decimal | 1010 = 10 decimal | 1110 = 14 decimal |
| 11 = 3 decimal | 111 = 7 decimal | 1011 = 11 decimal | 1111 = 15 decimal |

Nor was Leibniz's influence limited to calculating. The German philosopher developed many aspects related to logic. These were not discovered until after his death, since it would appear that he was not wholly satisfied with the results. In fact, he titled one of his works *Post tot logicas nondum logica qualem desidero scripta est*, which can be translated as *After So Many Logics, the Logic I Have Dreamed of Has Still Not Been Written*. His goal in terms of logic was to develop a universal system

## GOTTFRIED WILHELM LEIBNIZ (1646–1716)

The German thinker Gottfried Leibniz was one of the three great rationalists of the 17th century, together with Descartes and Spinoza. He was a mathematician, logician, philosopher, geologist, historian, and even an expert in jurisprudence; He also made great contributions to technology and anticipated concepts in biology, medicine, psychology and even computer science. He discovered infinitesimal calculus independently from Isaac Newton and his notation is what we use today. It is not possible to provide a comprehensive account of his achievements, since to the present day there is no full edition of his work, which is scattered throughout diaries, letters and manuscripts, some of which remain unpublished. Leibniz established a relationship between the binary representation system and the creation of the world. In his mathematical world view, which was reminiscent of Pythagoras, zero represented the void, and one God.

for calculation. He sought to develop a system for determining the inferences that were valid from a logical point of view. Hence, the calculus of logic could then be applied to arbitrary sciences. In one of his works, he wrote,

> "If this is done, whenever controversies arise, there will be no more need for arguing among two philosophers than among two mathematicians. For it will suffice to take the pen into the hand and sit down by the abacus, saying to each other (and if they wish also to a friend called for help): Let us calculate."

The influence of Ramon Llull's work is clear in these ideas. In fact, Leibniz's *Dissertatio de Arte Combinatoria* was inspired by Llull's *Ars Magna*. For the German, even the approximation of divine knowledge needed to be derived from a combination of basic concepts. These undefinable basic concepts should be expressed mathematically and used to arrive at correct propositions using clear deductive rules. Either way, the concepts and procedures form the basis of mathematical logic. Leibniz believed that there was a close relationship between logic, mathematics and metaphysics. He maintained that, "My metaphysics is entirely mathematical." and that, "I see that metaphysical truth is scarcely different from logical truth."

## New expressions for calculating the number π

Throughout the 17th century, a wide range of researchers took up the task of calculating $\pi$, picking up where François Viète had left off, following the same line of investigation. One of these was the Englishman John Wallis (1616–1703), from the University of Oxford. In his work *Arithmetica Infinitorum*, published in 1655, Wallis described a number of expressions for integrals and used these to obtain the expression for the number $\pi$, shown below:

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \frac{10}{9} \cdots$$

Based on the manipulation of this expression, the mathematician and philosopher William Brouncker (1620–1684), a founding member and first president of the Royal Society, derived the following formula in 1658:

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \ldots}}}}}$$

The next important expression arrived at in Europe came originally from beyond its borders. It was discovered by Madhava of Sangamagrama before being rediscovered by Leibniz in 1671, based on James Gregory's expression for the arctangent. Remember that its terms were:

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \ldots + (-1)^n / (2n + 1) + \ldots,$$

and the arctangent function was given by the following expression:

$$\arctan x = x - (x^3) / 3 + (x^5) / 5 - (x^7) / 7 + \ldots$$

## The 18th century

The 18th century has been historically referred to as the Enlightenment and the Age of Reason. It is not the purpose of this book to call the Enlightenment into question. However, in terms of computing, no progress was made on the advances of the previous century. Perhaps the science of the 17th century had been so brilliant that humankind had been blinded by all the new sources of light that spread out before it, and perhaps it had its work cut out studying and considering what had been illuminated. Regardless, computation, logic and the calculation of the number $\pi$ continued throughout those years along the same lines as in the 17th century.

## The calculation of $\pi$ in the 18th century

The 18th century provided some new expressions for calculating the number $\pi$. The first of these was the result of work by the astronomer John Machin (1680–1751), which maintained its supremacy for centuries. Machin studied the arctangent function and used the Gregory, Leibniz and Madhava formula to determine that the angle with an arctangent of 1/5 can be expressed as:

$$\alpha = \arctan\,(1/5) = (1/5) - ((1/5)^3)\,/\,3 + ((1/5)^5)\,/\,5 - ((1/5)^7)\,/\,7 + \ldots$$

Based on the arctangent of the angle $4\alpha - \pi/4$, he constructed a series for calculating the number $\pi$ using the inverse of the cotangent. In contrast to previous attempts, this series converged quickly, allowing the Englishman to efficiently calculate $\pi$ to 100 decimal places. The series corresponds to:

$$\pi\,/\,4 = 4\,\arctan\,(1/5) - \arctan\,(1/239).$$

This expression is described by the following series:

$$\frac{\pi}{4} = 4\left[\frac{1}{5} - \frac{1}{3\cdot5^3} + \frac{1}{5\cdot5^5} - \frac{1}{7\cdot5^7} + \ldots\right] - \left[\frac{1}{239} - \frac{1}{3\cdot239^3} + \frac{1}{5\cdot239^5} - \ldots\right].$$

Leonhard Euler also contributed to developing series for calculating the decimal places of the number $\pi$. He calculated $\pi$ to 20 decimal places in less than half an hour.

## LEONHARD EULER (1707–1783)

The Swiss mathematician and physicist Leonhard Euler spent the majority of his life in Russia and Germany. He is regarded as the most important mathematician of the 18th century and one of the greatest of all time. He made essential discoveries related to infinitesimal calculus and graph theory and also introduced a considerable part of modern mathematical terminology and notation, above all in the area of analysis, such as the notion of a mathematical function. He also made fundamental contributions to mechanics, fluid dynamics, optics and astronomy. An incredibly prolific author, it is calculated that his complete works could fill between 60 and 80 volumes.
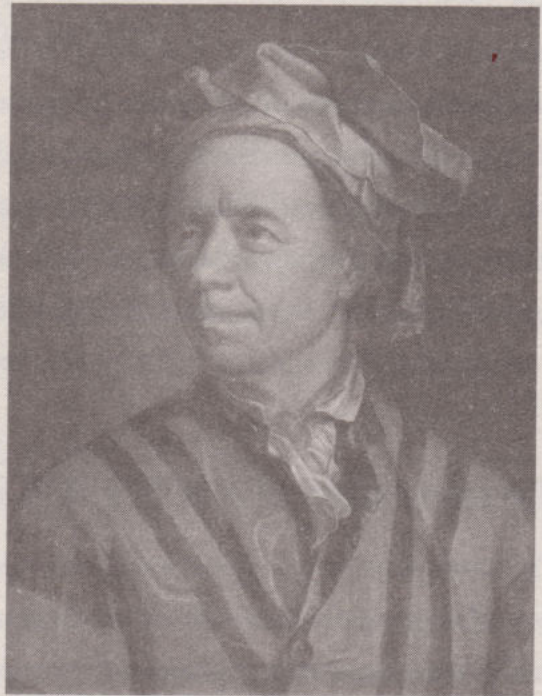
## THE NUMBER $\pi$

The use of the Greek letter *pi* ($\pi$) to denote the number $\pi$ was generalised by Leonhard Euler in his book *Introductio in analysim infinitorum*, published in 1748, where he uses it in conjunction with *periphéreia*, the Greek word for 'circumference'. Euler introduced other symbols that are extremely popular and widely used in current mathematics: The letter *e* for representing the base of the natural logarithm, the letter *i* for the square root of minus one, the symbol $\Sigma$ for the summation of a series, and $\Delta$ for a finite difference.

## Logic

Research during the 18th century did not make great progress in the realm of logic either. However there can be no doubting that the German philosopher Immanuel Kant (1724–1804), while not making direct contributions in this area, provided key elements for its subsequent development. In fact, Kant's ideas steered philosophy towards logical positivism and analytic philosophy. Years later, Frege, Hilbert, Russell and Gödel would go on to make the great contributions in the field of logic.

Immanuel Kant established the foundations of three of the main characteristics of modern logic: The distinction between concept and object; the primacy of the proposition as a unit of logical analysis; and the conception of logic for studying the structure of logical systems and not solely for validating individual inferences.

*A lecturer in logic and metaphysics at the University of Königsberg, the city in which he was born, Immanuel Kant is one of the key thinkers in the history of philosophy. His studies span a range of disciplines, such as law and aesthetics, and were especially important in the field of logic.*

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## THE DISTINCTION BETWEEN CONCEPT AND OBJECT

Gottlob Frege (1848–1925) established the fact that any sentence or proposition consists of an expression that denotes an object and a predicate that denotes a concept. Hence in the statement "Socrates is a philosopher," for example, "Socrates" is the object and "philosopher" the concept of being a philosopher. This point of view differed substantially from the prevailing logic at the time, which considered a proposition as two terms joined by the verb "to be." This new way of viewing the concept-object relationship was fundamental for understanding sets and the membership of their elements.

## The 19th century: some elements of calculation

The first calculator to be a commercial success was the Arithmometer, invented by the Frenchman Charles-Xavier Thomas de Colmar (1785–1870), which not only sold in great numbers in France but in other countries as well. Competition sprang up immediately, and a number of alternative models had been built within the space of a few years. The most significant were the *Arithmaurel* calculator, invented by the Frenchman Timoleon Maurel (1842); the pinwheel calculator, invented by the North American Frank Baldwin (1872), which was also developed independently by a Swedish inventor living in St. Petersburg, Russia, Willgot Odhner (1874); and the circular calculator developed by the Englishman Joseph Edmonson (1885). All these machines were still being used long into the 20th century.
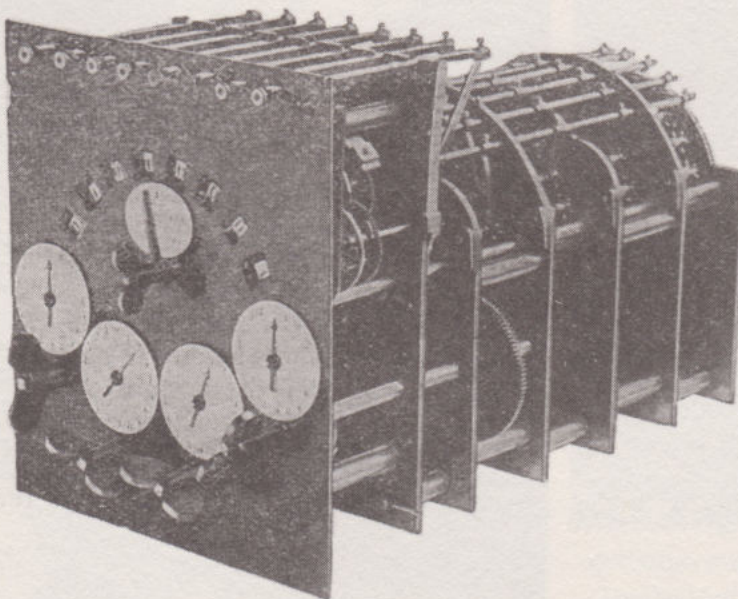


*The mechanism of an Arithmaurel, the calculator invented by Timoleon Maurel.*

From Maurel's machine onwards, calculators could find square roots in addition to the more basic arithmetic operations. The square root operation was based on the following series for $x^2$:

$$1 + 3 + 5 + ... + (2x - 1) = x^2.$$

Given a number $n$, which is a perfect square, the square root of $n$ can be calculated by successively subtracting 1, 3, 5… all the way down to zero. The number of subtraction operations carried out is the square root of the number. For example, if we consider the square root of 100, it is necessary to subtract 1, 3, 5, 7, 9, 11, 13, 15, 17, 19. Hence, since it has been necessary to subtract 10 numbers, the square root of 100 is 10.

If $n$ is not a perfect square, the final subtraction will give a negative number. The number of subtractions will give only an approximation of the square root. In order to determine the decimal points, we can proceed by multiplying by powers of 100 for each decimal we wish to obtain. For example, multiplying 2 by 100 to calculate the square root of 200, gives one decimal point. Hence:

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 =$$
$$= 196 < 200 < 225 =$$
$$= 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 + 29.$$

Note that there are 14 numbers in the top sum and 15 in the bottom. Hence, the square root of 200 lies between 14 and 15, and the square root of 2 between 1.4 and 1.5.

The 19th century saw the scientific advances that directly prepared the ground for our current computerised world. In 1835, the North American physicist Joseph Henry, known for his work on electromagnetism, invented the electromechanical relay, a decisive step towards computers. In addition to this, another much more specific advance, the appearance of the numeric keypad, was to prefigure an essential part of the interface that computers would have in the future. Up to this point, the methods used by calculators for entering numbers had multiplied the time for operations and also required training in calculation. The keypad made it possible to reduce the required operating time and placed the calculator within the reach of anybody.

With the steady advances in industrial technology that would culminate in the Industrial Revolution, automatic computing began to run in parallel to the process of automation in the textile industry. In 1725, the Frenchman Basile Bouchon had already started using techniques for programming looms with a perforated belt that contained information about the patterns. The belt was moved as the textile was woven. A few years later in 1728, his assistant Jean-Baptiste Falcon perfected the system and replaced the belt with a punched card system. In 1803 Joseph Marie Jacquard (1752–1834) developed a system based on the design of the engineer Jacques de Vaucanson, who had used cards and a rotating drum for the automatic production of textiles with a single operator in 1740. It nevertheless became known as the Jacquard loom. The punched card system, the most effective up to that point, continued to evolve and was still a well-trusted means of programming computers into the 1960s.

In a story now enshrined in the history of computing, the statistician Herman Hollerith (1860–1929) made use of the punched cards to encode the information for the American census in 1890. As such, he is regarded as the first informatics engineer – the first person to handle information automatically – and he also set up a company that would later be a foundation for International Business Machines, better known now as the computer giant IBM.
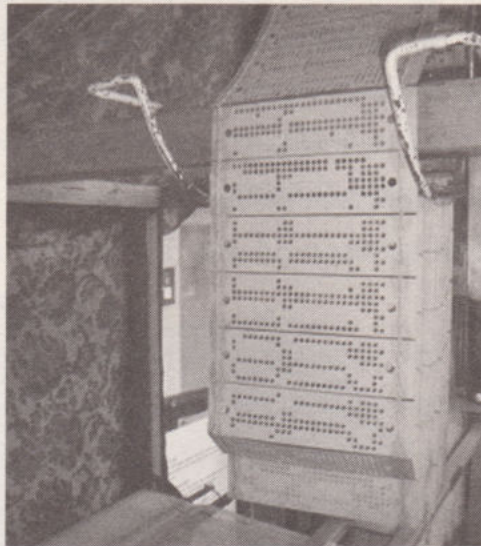
The punched cards on a Jacquard loom at the Manchester Museum of Science and Industry.

## Charles Babbage

The English philosopher, mathematician, inventor and mechanical engineer, Charles Babbage, is regarded as the father of computing and is one of the most brilliant and controversial figures in our story. It is believed he was born on the outskirts of Lon-

don in 1791, and we know for certain that he was baptised on 6 January, 1792, at St. Mary Newington. He studied mathematics and chemistry, first at Trinity College, Cambridge, which he entered in 1810, and then at Peterhouse (1812), a smaller and less prestigious college. It has been said that Babbage decided to change since two of his closest friends at Trinity College, astronomer John Herschel and mathematician George Peacock, were his intellectual superiors, whereas at Peterhouse he was able to graduate as top of the class in 1814, obtaining his masters in mathematics in 1817.



*Portrait of Charles Babbage by Samuel Laurence.*

In 1812 Babbage, Herschel, Peacock and other colleagues, under the supervision of the lecturer Robert Woodhouse, founded the Analytical Society with the aim of replacing the techniques of Newtonian calculus with those of Leibniz's analytical calculus. The society's two most relevant activities at that time were the translation from French of Sylvestre François Lacroix's *Traité de calcul différentiel et intégral* (*An Elementary Treatise on the Differential and Integral Calculus*; 1816) and Peacock's use of Leibniz's notation in his final exams (1817). Lacroix's three-volume book was translated by Babbage, Herschel and Peacock and was widely read in England. In 1819, the society changed its name to the Cambridge Philosophical Society, which still exists to this day.

In the same year as his graduation (1814), Babbage married Georgiana Whitmore, with whom he fathered eight children. When his father, his wife and at least one of his

children died in 1827, Babbage received an inheritance that included properties and a large sum of money, although he was nevertheless distraught from the loss. (Only three of his eight children survived.) Acting on the advice of his doctor, he travelled for a year around Europe. Upon his return, he took up the professorship at Cambridge, a post that had belonged to Newton. However, Babbage regarded the salary as low and only attended the University when it was necessary to assess the candidates for the Smith prize for the best student at Cambridge.

Charles Babbage has gone down in history as the designer of incredible calculating machines. The first of these was the Difference Engine, constructed by the British scientist with the aim of calculating the values of a polynomial. The design was based on finite differences in order to avoid multiplication and division. Babbage started building the machine in 1822 with funding from the British government. However the project was never completed. Its development was abandoned in 1834 when it lost its funding.
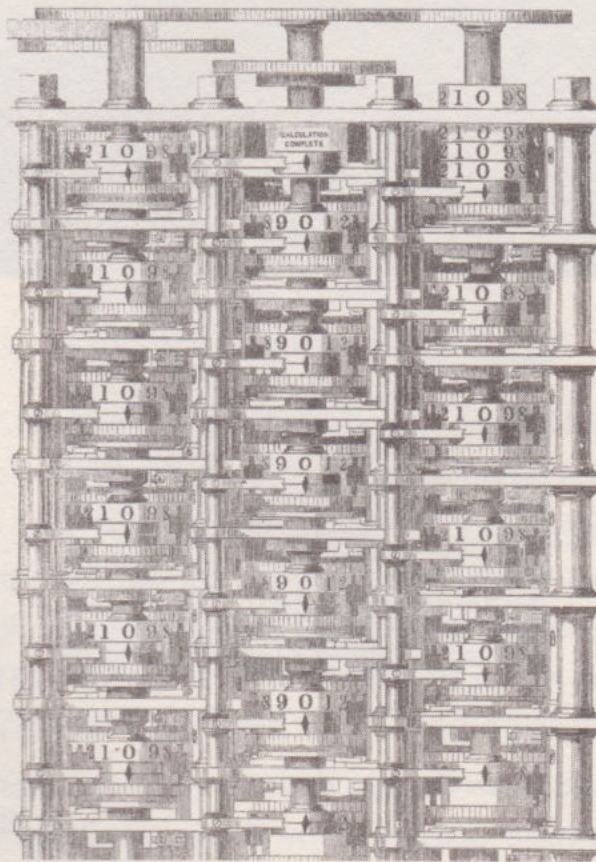


*Illustration of Charles Babbage's Difference Engine, as it appeared in* Harper's *magazine in December 1864.*

Some researchers have maintained that Babbage's machine could not be completed due to technical limitations prevailing at the time. However, Per Georg Scheutz (1785–1873) and his son Edvard, both from Sweden, built a machine in 1843 after reading an article on Babbage's Difference Engine. And later on, in 1851, with the help of the Swedish Academy of Sciences, they built a larger machine that was able to carry out calculations with a precision of fifteen decimal places and a system for printing the results.

Charles Babbage may not have been able to complete his machine, but it did work. In 1991 the Science Museum in London completed the inventor's first prototype using technology from the time. A second prototype has also been built, which is held at the Computer History Museum in Mountain View, California. Babbage's machine had a precision of 31 digits, was able to calculate degree seven polynomials and measured 2.4 $\times$ 2.1 $\times$ 0.9 metres. While the Scheutz's machine only measured 54 $\times$ 86 $\times$ 65 centimetres, the most advanced model was only able to calculate degree three polynomials, with a precision of 15 digits. In 2000, the Science Museum also built the printer that Babbage had designed for his machine.

After abandoning his Difference Engine in 1834, Babbage turned his attention to designing a new device, named the Analytical Engine, the most direct predecessor of modern computers. While the Difference Engine was only able to calculate polynomials, the analytical engine was designed to be general purpose, or rather to be able to calculate arbitrary functions. Babbage's new machine was to be powered by a steam engine, it took punched cards as an input, and had a printer and another system to punch new cards as its output. It also had a memory capable of storing 1,000 numbers with 50 digits (decimals), and an arithmetic unit that supported the four basic operations, which Babbage called the mill. The machine was programmed using a specific language that prefigured modern assembly code. In addition to the basic instructions for operations, the language implemented loops, conditionals and storage. From a formal and mathematical point of view, the set of possible operations envisaged by Babbage for his project would have a power, or calculating capacity, equivalent to a Turing machine, although it was comparatively slower in terms of processing time.

Babbage had a very special collaborator who helped him develop the second machine, Ada Lovelace, daughter of the poet Lord Byron. Her contributions were late in being recognised, but despite this, Ada Lovelace is now regarded as the first computer programmer. As well as writing programmes in the programming language

## ADA BYRON, COUNTESS LOVELACE (1815–1852)

Ada Augusta Byron was the only child of Lord Byron and his wife Annabella Milbanke. She never knew the poet, since her parents separated one month after her birth, and Lord Byron left England never to return. Her sickly nature, inherited from her father, meant that she was educated at home, with a special emphasis on mathematics and sciences, by famous teachers such as William Frend, William King, Mary Somerville and Augustus de Morgan. Her teachers believed she had the potential to be a first-class researcher, and in fact it was Mary Somerville who introduced her to Charles Babbage. In recognition of her pioneering work in programming and creating computing languages, in 1979 the United States Department of Defense named their new programming language Ada.

for the Analytical Engine, she deduced and envisaged computers that would go beyond the simple numerical calculations on which Babbage was solely focused.

The history of this partnership began when Babbage asked the then Ada Byron to translate a text written in French by Luigi Menabrea on the Analytical Engine, based on a lecture given by Babbage in Turin upon invitation by the mathematician Giovanni Plana. Ada added a series of notes to the translation, which were in fact more extensive than Menabrea's report. The famous note G describes how to calculate Bernoulli numbers using the programming language for Babbage's machine, with two loops to demonstrate the machine's forking capacity. It was the first computer programme in history. Ada also described how to carry out trigonometric operations with variables.

Some researchers have raised doubts about the authorship of note G. Was it perhaps written by Babbage himself? Regardless, Ada's mathematical knowledge and familiarity with the Analytical Engine are indisputable, and her partnership with its inventor was so close that it is impossible to say to what extent she influenced

## THE FUTURE IN NOTE G

In note G, Ada Lovelace expresses her confidence that, not only Babbage's machine, but the new way of handling information will constitute a scientific revolution: "The Analytical Engine has no pretensions whatever to originate anything. It can do whatever we know how to order it to perform. It can follow analysis; but it has no power of anticipating any analytical relations or truths. Its province is to assist us in making available what we are already acquainted with. This it is calculated to effect primarily and chiefly of course, through its executive faculties; but it is likely to exert an indirect and reciprocal influence on science itself in another manner. For, in so distributing and combining the truths and the formulae of analysis, that they may become most easily and rapidly amenable to the mechanical combinations of the engine, the relations and the nature of many subjects in that science are necessarily thrown into new lights, and more profoundly investigated. This is a decidedly indirect, and a somewhat speculative, consequence of such an invention. It is however pretty evident, on general principles, that in devising for mathematical truths a new form in which to record and throw themselves out for actual use, views are likely to be induced, which should again react on the more theoretical phase of the subject. There are in all extensions of human power, or additions to human knowledge, various collateral influences, besides the main and primary object attained."

its design. In fact, Ada had expert knowledge of the Jacquard loom system, and there are authors, in some instances the same ones as those mentioned above, who believed she was the source of the idea to use punched cards on the machine for inputting programmes and data. Ada developed concepts that are now so familiar in programming languages, such as instructions, loops and subroutines. In fact, on account of her contributions, Babbage called her the Enchantress of Numbers.

Once more, it was not possible to construct the machine, this time due to financial, political and even legal problems. Only certain sections were built, such as parts of the arithmetic unit and the printing system. Neither the memory nor any programmable part was developed. Computers that were logically comparable would not be built for another one hundred years. The Analytical Engine was forgotten by the world, except for a number of inventors, whose achievements bear the mark of its powerful influence.

In 1903, the Irish accountant Percy Ludgate designed a machine with similar features to that of Babbage, although it was now powered by electricity instead of steam. Leonardo Torres y Quevedo, a Spanish engineer and mathematician, and

prolific inventor, used those ideas to create his Chess Player machine (El Ajedrecista) in 1911, which played a king and rook against the king of a human opponent in a somewhat imprecise manner, but always using the minimum number of movements – and winning of course.

Later on, in the 1930s, the North American scientist Vannevar Bush developed a number of machines for solving differential equations and designed a digital electronic computer. Even the first electromechanical computer, the Harvard Mark I, developed between 1939 and 1943 by the North American engineer Howard Hathaway Aiken, with support from IBM, based its 760,000 wheels and 800 kilometres of cable on notions developed by Babbage's machine.

If it had been built, the Analytical Engine would have measured 30 metres long by 10 metres wide and would have been 4.5 metres tall. Addition would have taken 3 seconds and multiplication between 2 and 4 minutes, assuming the data had already been loaded onto the arithmetic unit. It would have taken 2.5 seconds for the numbers to reach the arithmetic unit.

Charles Babbage is known for many other discoveries and contributions. He cracked the Vigenère code, a variation of the code used by Julius Caesar. He invented the fender for train engines, which removed obstacles on the track, and even left a legacy in economics, describing what is still known as the 'Babbage Principle'. He also proposed the postal franking system that is used to this day and was the first to observe that the width of the ring of a tree depended on the weather conditions for that year, making it possible to deduce historical climate data by studying the oldest trees.

However, when it came to philosophy and theology, two fields he also wished to excel in, he was not so brilliant. He was a faithful Christian, and in 1837 he published the *Ninth Bridgewater Treatise*, which continued the eight treatises on natural theology financed by the legacy bequeathed by Rev. Francis Henry, Count of Bridgewater, that had been published to that date. Babbage contributed to the clergyman's justification of the existence of God from the perspective of automated mathematics. He wrote that God, as a divine legislator, wrote laws or programmes that produced life on Earth independently, instead of the Almighty creating them directly. He also used mathematics to justify miracles by calculating their probabilities. The text was contemporaneous with the work of Charles Darwin (1809–1882).

# Logic and George Boole

In 1847, George Boole published his book *Mathematical Analysis of Logic,* in which he defined what is now known as Boolean algebra, an attempt to use algebraic techniques to handle propositional logic expressions (first order logic), and later predicate logic. Boolean logic is currently generally applied to electronic design, however at the outset Boole's discoveries were met with limited acknowledgement among experts in the field. Prior to the 20th century, awareness of their importance or their use in computing remained poor.

The person largely responsible for their rehabilitation was Claude Shannon (1916–2001), the North American engineer and mathematician who is regarded as the father of information theory. Shannon was aware of Boole's work from his philosophy classes at the University of Michigan. In 1937 he wrote his masters thesis at the Massachusetts Institute of Technology (MIT), showing that Boolean algebra could be used for optimising circuits. In 1935, in parallel to the work of Shannon, but independently, the logician Víctor Shestakov (1907–1987) also used Boolean logic for the same purpose at the Moscow State University.

Boolean logic is so useful for computing because it provides the ideal circumstances for the development of a binary logic. It operates on the values zero and one, which it handles using the basic operations AND, OR and NOT, or conjunction (binary operation denoted by $\wedge$), disjunction (binary operation denoted by $\vee$) and negation (unary operation denoted by $\neg$), although there are also more complex properties and operations. The operations and their results can be defined using truth tables:

| x | y | $x \wedge y$ | $x \vee y$ | $\neg x$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 0 |

The standard operations, such as implication, are expressed using the three previous operations $(x \rightarrow y) = \neg x \vee y$. Furthermore, any other function that can be constructed from the inputs can be expressed as a combination of the basic operations. In fact, the so-called De Morgan's Law ensures that there can only be two basic operations. For example, conjunction can be expressed in terms of disjunction and negation.

## GEORGE BOOLE (1815–1864)

The British mathematician and philosopher George Boole is regarded as one of the founding figures of computer science, largely thanks to his development of the algebra that would form the foundations of modern computation. His most important mathematical works were *Treatise on Differential Equations* (1859), and its continuation, *Treatise on the Calculus of Finite Differences* (1860). However, it was his work on logic, published under the long and wordy title *An Investigation of the Laws of Thought, on Which are Founded the Mathematical Theories of Logic and Probabilities*, in which he developed his system of rules for expressing, handling and simplifying logical and philosophical problems. The system employed mathematical procedures with arguments that permitted just two states: true or false.
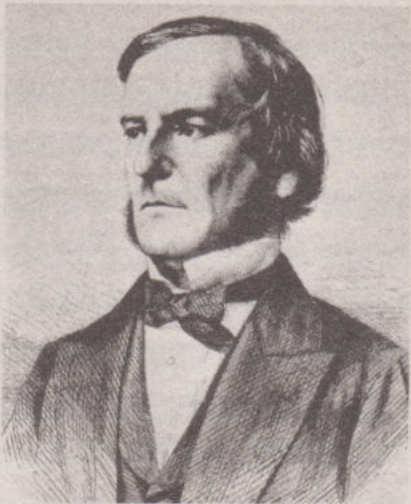
The axioms for Boolean logic are provided by properties, which informally ensures they are necessary and sufficient to construct the truth tables.

| Associativity | $x \vee (y \vee z) = (x \vee y) \vee z$ | $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ |
|---|---|---|
| Commutativity | $x \vee y = y \vee x$ | $x \wedge y = y \wedge x$ |
| Absorption | $x \vee (x \vee y) = x$ | $x \wedge (x \vee y) = x$ |
| Complements | $x \vee \neg x = 1$ | $x \wedge \neg x = 0$ |
| Distribution | $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ | |

## The number $\pi$ in the 19th century

In the second half of the 18th century, in 1761, Johann Lambert (1728–1777), the French-born German astronomer and philosopher, had shown that both the number $\pi$ and its square $\pi^2$ were irrational, hence discounting the possibility of being able to determine an 'exact' figure for the number. The calculation of the number $\pi$ would have to wait 120 years before experiencing its next mathematical breakthrough. This occurred in 1882, when the mathematician Ferdinand Lindemann (1852–1939)

showed that the number π was a transcendental number, which led him to resolve the problem of squaring the circle by proving it was impossible.

There are currently still open mathematical problems related to the number π, such as the normality of π. A number is referred to as irrational when the frequency of all sequences of numbers of the same length have the same possibility of appearing; for example, all digits appear with probability 1/10, all sequences of two numbers appear with probability 1/100, etc. It has not been possible to prove that π is a normal number even though this is believed to be the case and despite frequency analyses having been carried out to support this theory. At the end of the 20th century, the North American mathematician David Bailey carried out a study of the first 29,360,000 decimal places, considering sequences of up to 6 digits without finding any irregularity. The differences between the frequencies are minor and are not statistically significant. For example, the frequencies of the digits 0–9 are given below, according to David Bailey's work:

| Digit | Frequency |
|:-----:|:---------:|
| 0 | 2,935,072 |
| 1 | 2,936,516 |
| 2 | 2,936,843 |
| 3 | 2,935,205 |
| 4 | 2,938,787 |
| 5 | 2,936,197 |
| 6 | 2,935,504 |
| 7 | 2,934,083 |
| 8 | 2,935,698 |
| 9 | 2,936,095 |

## ALGEBRAIC NUMBERS AND TRANSCENDENTAL NUMBERS

A number is algebraic when it can be expressed as the solution of a single variable polynomial with integer coefficients. All integers and rational numbers are algebraic, as well as certain irrational numbers. The best known irrational number is the square root of two: $\sqrt{2}$, which is given by the solution to the polynomial $x^2 - 2 = 0$. The set of algebraic numbers is countable. A number is transcendental when it is not algebraic, i.e. it cannot be written as the solution of a polynomial with integer coefficients. The best known examples are π and the number e.

# Chapter 4

# Hardware in the 20th Century

The 20th century was regularly convulsed by all sorts of political and social changes, and this was also true in the realms of thought and science, not least the spectacular revolution in technology. Against the backdrop of the century's great scientific breakthrough, incredible human achievements and nadirs of cruelty and destruction, computing steadily re-shaped society. The power of the computing revolution has been staggering, but the underlying architecture of the machines has not changed much. To this day, computers continue to follow the basic architecture devised by mathematician John von Neumann in the 1940s.

## Konrad Zuse's Z Series

The history of computing and computers in the 20th century has quite a number of headliners. One of the first was largely forgotten about or simply ignored for many years – the German Konrad Zuse, who built the Z Series machines. The majority of Zuse's innovations went unnoticed for many years after, largely because they were built in the lead up to the World War II and were developed throughout the conflict. In creating his design Zuse followed the structures of Charles Babbage, albeit unwittingly, since he was unaware of the Englishman's work. Similarly, later on, when John von Neumann described his structure, he was also unaware of the work of Zuse. All three arrived at what is without doubt the most logical structure, comprised of a control system, a memory and the corresponding arithmetic unit for carrying out calculations.

Zuse built his first machines between 1935 and 1939, when Europe was inexorably approaching that tipping point that would lower it into the hellish Second World War. Zuse's first patent request is dated 11 April, 1936. In 1938, he requested a patent in the United States, which was rejected due to the lack of detail. The machines were named the Z1 and Z2. Zuse first considered the name V1, for *Versuchsmodell*, which means 'experimental model', but changed the name to avoid confusion with the V1 and V2 rockets developed by Werner von Braun.

The dimensions of the Z1, the first machine, were 2 × 1.5 metres. The machine was built from steel and operated with frequent difficulties. In fact, it was nothing more than a mechanical binary calculator, which operated using electricity and with a limited programming capacity. The German built the Z2 to solve the problems of the first machine. This second machine employed relay switching for the memory and fixed point numbers.

On the suggestion of Helmut Schreyer, Zuse later considered replacing relays with vacuum tubes and designed the fully functioning Z3. It was unveiled on 5 December 1941 at the German Institute for Aeronautical Research. Data was input into the Z3 machine using a keypad, and the control programme was read from a celluloid strip. It had a memory for 64 numbers represented in binary with a floating point. Each number was represented using a total of 22 bits. One sign bit, seven bits for the exponent, and 14 for the coefficient. Zuse discovered that by using floating point representation, the first bit could always be equal to 1. It was only necessary to use the correct exponent. This is the representation that is still currently used, because it makes it possible to eliminate the bit that always has the value of one. In the case of the Z3 this meant the machine could represent numbers as if the coefficient were 15 bits. However, the machine had one important limitation: It did not allow conditional jumps. In terms of its speed, it was able to carry out three or four additions per second, and could multiply two numbers in four or five seconds.

Zuse immediately began to design and build the Z4 as an assignment for the German Institute of Aeronautical Research. The machine was unveiled on 28 April, 1945. It had a memory of 1,024 numbers, each with 32 bits, and implemented conditional jumps and subroutines. It also implemented a mechanism that read two instructions in advance, making it possible to insert an operation if they did not modify the result, delivering a performance increase in terms of time. This process was also common in subsequent computers and has become known as *lookahead*.

As the war drew to an end, Zuse moved to a farm in the Alps, where he began to write his PhD thesis: *Theory of General Computation*. In his text, completed in 1946, he defined *Plankalkül* (the calculation of planes) by making reference to a theoretical programming language, which was never put into practice. The *Plankalkül* language was designed to solve both numerical and non-numerical problems and had a high level of abstraction, far superior to similar attempts at the time. The first real programming language that was comparable was Algol, which was developed some years later.

## KONRAD ZUSE (1910–1995)

While he was studying at university, the German engineer Konrad Zuse had to make so many routine calculations by hand that he grew bored and began to dream of a machine that could help him. Upon completion of his degree, and after a stint at an aircraft factory, he left his job to follow his dream and built his first machine in his parent's apartment. It was not long before he became the creator of the first fully-working programmable computer. He built the mythical general-purpose Z Series; the Z1 and Z2, machines for carrying out the calculations needed for dropping bombs, and the L1 machine for evaluating logic functions. He also defined the *Plankalkül* programming language, albeit only theoretically. He founded a number of companies for building his machines, the most important of which was Zuse KG, which further developed members of the Z Series and is regarded as the first ever computer company.

From 1947 onwards, in the devastating aftermath of the war, Zuse returned to his work with machines. He was in contact with IBM and later with Remington Rand, with whom he signed a contract. He developed vacuum-tube-based machines, such as the Z22, and machines that used transistors, such as the Z23 and a remodelled Z3. Later, he built the Z64, a machine-controlled plotter.

*The only surviving example of these early Zuse machines is the Z4, which* became the first commercially available computer. The machine was employed until 1959 by a range of institutions. There is one on show at the Deutsches Museum in Munich, Germany, together with a replica of the Z3. Unfortunately, all the other machines were destroyed during the wartime bombing of Berlin.

## The Turing machine and Colossus

Despite wishing to be a doctor as a child, Alan Turing (1912–1954) became a mathematician, philosopher and cryptographer, and another of the forefathers of computing. He is known above all for his theories, but he also played an important role in the practical construction of one of the first modern computers.

Turing's success began on the more theoretical plane of mathematics in 1936, when he solved the decidability problem, referred to using its German name, *Entscheidungsproblem*, as formulated by David Hilbert. His solution was based on designing a model for calculation that formalised the concept of an algorithm (or programme) and which history has christened the Turing machine.

The mathematician Alan Turing, regarded as one of the founding figures of computing.

In 1928 the influential German mathematician David Hilbert (1862–1943), who had challenged the world's mathematicians with a series of famous problems in 1900, returned to the problem of decidability, which dated back to Leibniz. In his opinion, there were no unsolvable problems, and he proposed the hypothesis that it is always possible to devise a programme (or algorithm), which when given a description of a question tells us whether it is true or false. Independently, Alan Turing and his American colleague Alonzo Church proved that Hilbert was wrong. There are unsolvable problems and it is not possible to devise the algorithm that Hilbert described. As such, mathematics is not decidable, or rather, there is no single defined method that can be applied to any mathematical statement to establish whether it is true or false.

Church and Turing used their own models in their respective proofs. The former used the lambda calculus, and the latter, his now famous 'machine'. Both formally stated the concept of an algorithm and based their proofs on questions about arithmetic. If it was possible to prove there was no solution for this type of question, then it would be enough to generalise it for the whole of mathematics. Turing's work was the more accessible and intuitive. The British scientist reduced the problem of decidability to the problem of termination, showing that it was unsolvable for his

theoretical machine. It is not possible to algorithmically decide whether a Turing machine will terminate. Nevertheless, the two great logicians did not become rivals; quite the reverse. Realising that their models, despite being formally different, were equivalent, they joined forces.

## HOW DOES THE TURING MACHINE WORK?

Imagine an infinite strip, which holds the input symbols for a problem and which can be written on with ease. The Turing machine has a reader located at one position on the strip. This reader makes it possible to read from and write to the strip, and the machine's programme allows it to move the strip by one position at a time. The possible states of the machine are represented by a set of states Q, and the machine's programming is represented by a function referred to as the 'transition function,' which defines the new state the machine will enter based on the current state and input symbol.

Formally, a Turing machine is defined as a tuple comprising seven elements. A tuple is an ordered sequence of objects, or rather a list with a limited number of objects referred to as a 'family'. Tuples are used to describe structured mathematical objects. We can denote the tuple to be processed by our Turing machine as

$$TM = (\Gamma, \Sigma, b, Q, q_0, f, F).$$

Its elements are defined as follows:

- $\Gamma$: the alphabet of the symbols on the strip.
- $\Sigma \subset \Gamma$: the alphabet of the input symbols. The set of symbols that can form the input set is a subset of those that can appear on the strip. The strip will also contain the symbols written by the machine.
- $b \in \Gamma, b \notin \Sigma$: $b$ represents the blank space. This symbol does not belong to the input set. To begin, the strip will contain a finite number of symbols from $\Sigma$, and the other positions (remember it is infinite) will contain the symbol $b$.
- $Q$: the set of states.
- $q_0 \in Q$: the start state.
- $f$: the transition function. Given a state and an element on the strip, this function defines a new state, writes a symbol on the strip and moves the position of the reader on the strip to the left (L), right (R) or keeps it in the same position (S, for stopped). Hence, $f$ is a function such that $f: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$.
- $F \subseteq Q$: the set of end states.

In April 1936, Alonzo Church, from Princeton University, had published a work on the problem of decidability. Church came to the same conclusion as Turing and showed that not everything is computable. In order to do so, he made use of the lambda calculus, which was radically different from the Turing machine and which he did in partnership with his colleague Stephen Kleene. Turing published his solution to the decidability problem shortly after, in August 1936. It was this second work, titled *On Computable Numbers, With an Application to the Entscheidungsproblem*, which reformulated the results of Kurt Gödel (1906–1978) on the limits of provability and computation, and even made reference to Church's early paper. However, instead of fighting for the glory of their mutual discovery, Turing moved to Princeton in September, 1936, to write his thesis on decidability under the supervision of Church. He presented it in 1938, obtaining his doctorate before returning to Cambridge.

## BLETCHLEY PARK AND ENIGMA

The famous military facility, Bletchley Park, was located in Buckinghamshire, 80 kilometres from London, between Cambridge and Oxford. Britain's most eminent scientists worked there, deciphering the German codes used during the Second World War. It took its name from the Victorian mansion that formed the hub of the complex. That space currently houses a museum of cryptography. The code of the Enigma machine was deciphered at Bletchley Park. The machine had a rotating cipher mechanism for encoding and decoding messages used by the armed forces of Nazi Germany. The British efforts to break the code were not in vain because reading messages that were assumed by the Germans to be totally secure hastened the end of the war in Europe by as much as two years.

These decisive events occurring on the high, sparsely populated planes of theoretical mathematics were taking place at one of the most tempestuous periods of human history. The political situation in Europe was increasingly tense, and the drums of war had started to beat. Mindful that Britain would be likely to declare war on Germany, Turing turned to the study of cryptography as he prepared his doctoral thesis. In 1939, the British government recruited him to work at Bletchley Park, together with other researchers, with the mission of breaking the secret code used by the German military, the Enigma Code. Turing's knowledge allowed him to decipher the code used by the German air force in the second half of 1941. In February, 1942, the Germans made the code more complex and once again the Allies were unable to decipher it.

Turing and his colleagues constructed a number of calculating machines to break the German code. Turing had already built a machine for multiplication during his stay at Princeton, but the requirement to increase the speed of calculations led the Turing group to develop the so-called Colossus machine, regarded as the world's first digital programmable electronic computer. Ten machines were built. The first of these began operation in December 1943, two years after ENIAC in the United States, which we shall discuss later. At the same time, towards the end of 1942 and the start of 1943, Turing travelled to the United States again, this time to advise the Americans on deciphering German codes. On the journey he met Claude Shannon, the founder of information theory and responsible for the definition of entropy.

After the war, Turing was contracted by the National Physical Laboratory (NPL), which develops science and technology standards in the United Kingdom. There, he worked on the design of the Automatic Computer Machine (ACM), a general purpose machine, and devised the concept now known as microprogramming, a term which refers to the use of a pre-programmed arithmetic unit. The alternative is for the unit's function to be directly implemented in the hardware, meaning that its operations cannot be modified. At that time, Turing also developed the concept of subroutines and software libraries.

In 1947, Turing took a year out in Cambridge, during which he wrote a report describing what are now referred to as neural networks, a computing model based on the physiology of nervous systems. The following year, seeing that the NPL machine was not making progress, he moved to the University of Manchester, where he worked on the development of Maxwell Newman's Mark I software. It was during this period he completed his most abstract studies. He wrote the famous article *Computer Machinery and Intelligence*, published in *Mind* magazine in October 1950, which set out his point

of view on artificial intelligence and proposed the experiment known as the 'Turing Test' to prove the existence of intelligence in a machine and decide its 'sentience'.
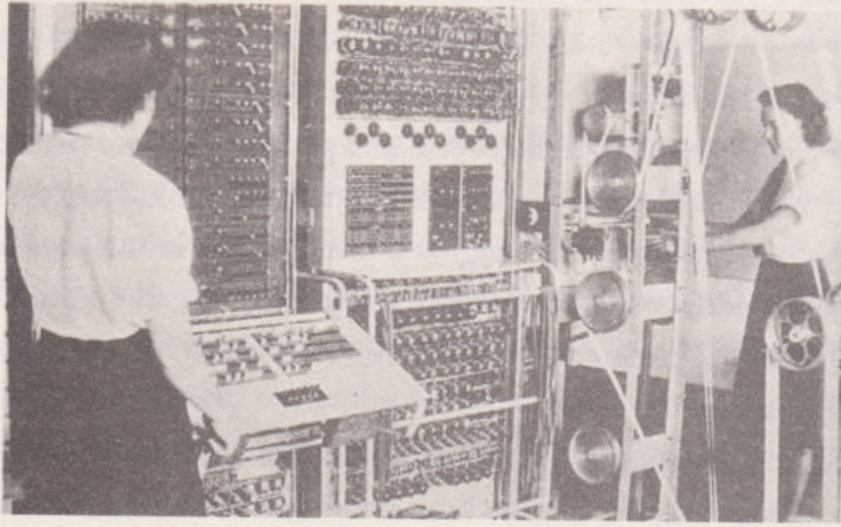


*Two operators working the Mark 2 version of Colossus.*

## SHAME AND INJUSTICE

In his final years, Alan Turing the mathematician and logician who invented much of the modern world suffered persecution due to the prejudices of post-war Britain. Many suggest this led to him claiming his own life. Alan Turing's brilliant career was cut short when he was tried for homosexuality, still illegal at that time in the United Kingdom, on the same charges brought against Oscar Wilde more than 50 years before. Turing knew he had nothing to be ashamed of and submitted to widely publicised court proceedings. The court gave him the option of a jail sentence or undergoing a chemical treatment to reduce his libido. He wished to avoid jail, however a year of oestrogen treatment resulted in significant physical alterations and rendered him impotent. He then committed suicide by cyanide. The story is more dramatic still when we consider that in his letters to his colleagues, his greatest concern was always that the personal attacks might overshadow his theories on artificial intelligence. In 2009, the then prime minister of the United Kingdom, Gordon Brown, issued a statement apologising in the name of the British government for the treatment received by Alan Turing during the final years of his life. It was a retraction that came fifty years too late for a man who had also provided a service of incalculable value to his country and the civilised world as a whole in the fight against the Nazis.

Turing was admitted to the Royal Society in 1951. However, he died without receiving the full recognition he deserved. Years after his tragic death, in 1966, the American Association for Computing Machinery created a prize in his honour, the Turing Prize, which is the equivalent of the Nobel Prize in the world of computing. The prize is awarded to distinguished researchers for their achievements in hardware and software, databases, the theoretical foundations of computing (including cryptography) and networking.

## The von Neumann architecture

John von Neumann's entry into the history of computing also took place in the context of the Second World War, while he was working on the Manhattan Project to develop the first atomic bomb. The project's requirements meant that powerful calculating machines had to be used, at a time when advanced and functional machines had only just begun to appear. Von Neumann scoured the United States for the most distinguished researchers in the field of computing. These included John Presper Eckert (1919–1995) and John William Mauchly (1907–1980), who had built the ENIAC system (Electronic Numerical Integrator and Computer) at the University of Pennsylvania, the American response to Britain's Colossus. The researchers set out to build a new and improved machine, EDVAC (Electronic Discrete Variable Automatic Computer). In March 1945 von Neumann formalised his ideas in the now famous document *First Draft of a Report on the EDVAC*, which he published under his sole name, causing a bust-up with the other two researchers involved.

The *First Draft* described what is now known as the von Neumann architecture, the most efficient structure of a computer. The structure is based around the internal storage of programmes, placing them in the computer's memory, and the separation of processing and storage units. Von Neumann also recognised the efficiency of having the same memory device for programmes and data. The general procedure of the von Neumann system follows three stages:

1. Recover the instruction from memory.
2. Decode.
3. Execute.

## ENIAC

ENIAC was completed and unveiled to the press in 1945 and was finally switched off ten years later, in 1955. The team behind its construction was made up of Herman Heine Goldstine, John Presper Eckert and John William Mauchly. Its spectacular frame took up 63 square metres, and weighed 30 tons; it was 2.6 metres high, 0.9 metres wide and 26 metres long. The machine was made up of 18,000 vacuum tubes, 72,000 diodes, 70,000 resistors and 1,500 relays. Its construction required 5 million solder points and cost almost half a million dollars at the time. It had no memory, instead relying on 20 accumulators that allowed it to store 20 numbers, each with 10 digits. Furthermore, switches made it possible to save the values of functions (104 values with 12 digits). It took 0.2 milliseconds to calculate addition operations and 2.8 milliseconds for multiplication. ENIAC heated the room to 50°C and there was a rumour (perhaps a malicious one) that when it was switched on, it caused blackouts in the city of Philadelphia, where it was located, since it consumed some 160 kW of power.
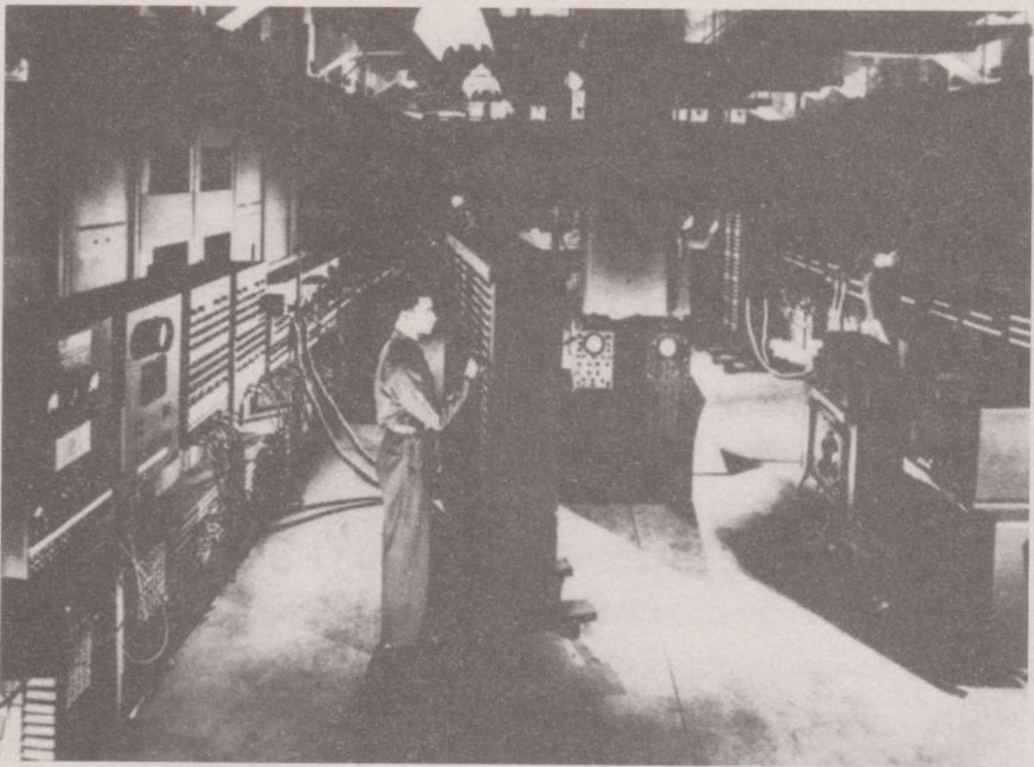


The process is sequentially applied according to the way the instructions are ordered in the memory, except when there is an instruction for an unconditional jump. This architecture does not differ greatly from the designs used by Charles Babbage and Konrad Zuse and is the same one on which the structure of our current computers is based.

## JOHN VON NEUMANN (1903–1957)

The Hungarian-American John von Neumann was one of the most important scientists of the 20th century. A pioneer of modern digital computing, his achievements were felt in many other areas: quantum physics, cybernetics, economics and, of course, mathematics. He was a child prodigy, and in 1926 at the age of just 23, he obtained a doctorate in mathematics with his doctoral thesis on the axiomatisation of set theory. In Europe, he carried out research in the disciplines of mathematics and quantum physics, principally in Göttingen, under the supervision of David Hilbert, and wrote the German book the *Mathematical Foundations of Quantum Mechanics*. In 1930 he emigrated to the United States and at the age of 29 obtained one of the five professorships at Princeton's IAS, whose previous incumbents included Albert Einstein. He is regarded as the father of the MAD concept (Mutually Assured Destruction). He worked on the Manhattan Project and the later development of the hydrogen bomb.

After the publication of the *First Draft*, von Neumann sought funding to build a more powerful machine. This was ADIVAC, developed at the IAS (Institute for Advanced Study) at Princeton and which began operation in 1953.

## The first computers in the United States

As a general purpose computer, ENIAC represented significant progress, although some of its technological solutions left much to be desired. Its main problem was that it was not controlled by a programme, and the connections of its circuits had to be changed in order to reprogramme the machine. As there was no programme stored in the memory, unlike modern computers, specific switches had to be connected or disconnected by hand, just like in old-fashioned telephone switchboards. This was an arduous task and took so long that it negated any advantage of the machine's then record-breaking operational speeds.

The design of EDVAC was meant to eradicate the problems of ENIAC. Eckert, Mauchly and von Neumann gave serious thought to how a computer should be programmed and concluded it was necessary to use a programme that was not

hard-wired, but stored in the memory in a similar way to numbers or other data. Nowadays, it may seem odd, and perhaps even hard to understand, that at some primitive moment in the development of computers the programming (the software) and the configurations of the computer (the hardware) were not separate entities. The moment of this great breakthrough heralded the birth of programming.
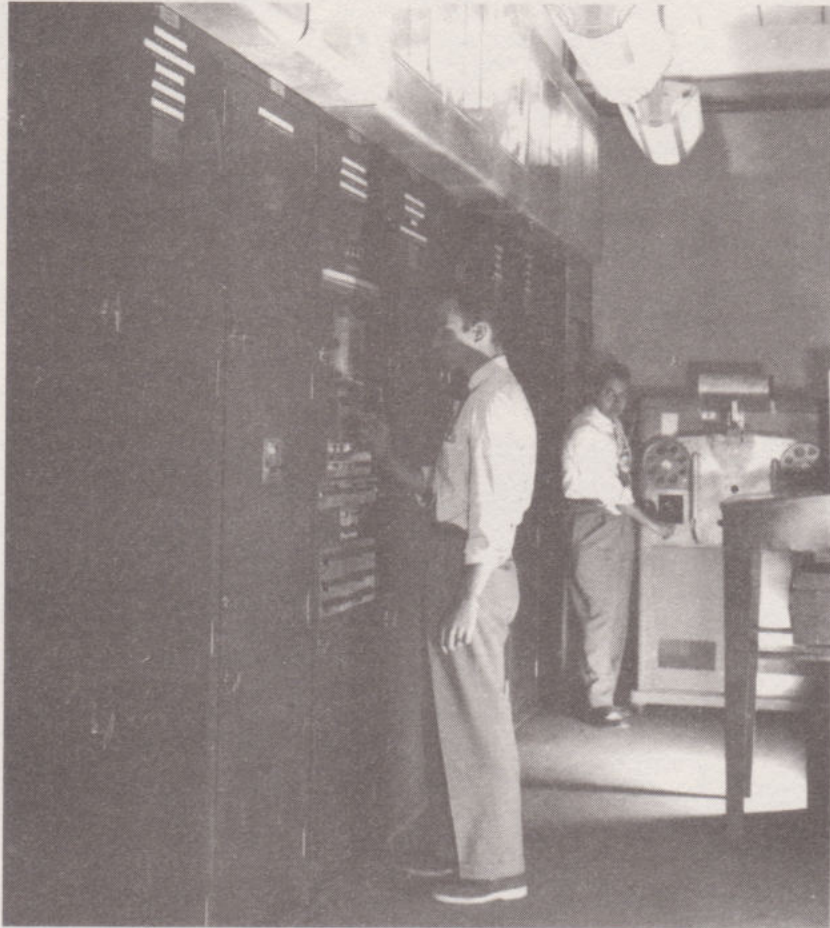


*EDVAC represented a great step forward in the history of computing.*

The machine was completed in 1949 at the Moore School of Electrical Engineering at the University of Pennsylvania. By then the system's designers Eckert and Mauchly had bowed out of the project. The machine had 6,000 vacuum tubes and 12,000 diodes. It weighed 7,850 kg and occupied 45.5 square metres. It needed 56 kW of power to run. It was nevertheless much lighter than ENIAC, if it is possible to make such a claim given those vital statistics. It also represented a marked performance increase in terms of time: Addition took 864 microseconds and multiplication 2,900 microseconds.

Meanwhile, the influence of von Neumann's *First Draft* was being felt in the United Kingdom. Maurice Wilkes from the University of Cambridge built EDSAC (Electronic Delay Storage Automatic Calculator), also completed in 1949. It was the first electronic calculating device with internal commands, although it was not the first computer to have internal programming, an honour that fell to SSEM. Its memory had 512 positions, each with 17 bits. The first video game in history, an electronic three-in-a-row game named OXO, was designed for EDSAC. The design of the machine provided the basis for LEO I, the first computer built purely for commercial purposes. EDSAC could also implement subroutines and even multiple subroutines.



## New Calculating Wizard

EDSAC, a British cousin of our electronic mathematical brains, such as ENIAC and EDVAC (PS, May '47, p. 95), will handle 10,000 multiplications a minute. Now under construction at England's Cambridge University, EDSAC will remember details of calculations and use "judgment" in choosing the best way to reach a result.

*A U.S. newspaper cutting reports the construction of EDSAC.*

Upon leaving the University of Pennsylvania, Eckert and Mauchly founded a company, the Eckert–Mauchly Computer Corporation, to build UNIVAC (Universal Automatic Computer). The word 'universal' meant that it was a general purpose machine that could solve problems related to science, engineering, economics etc.

The first customer to purchase Eckert and Mauchly's UNIVAC was the United States Census Office. By completion of the computer in 1951, the designers' company had been taken over by a larger firm, Remington Rand. The second machine was purchased by the Pentagon in 1952. These machines are regarded as the first commercial computers, and in the majority of cases they were used more for processing a great bulk of data, than for calculations related to physics or mathematics.
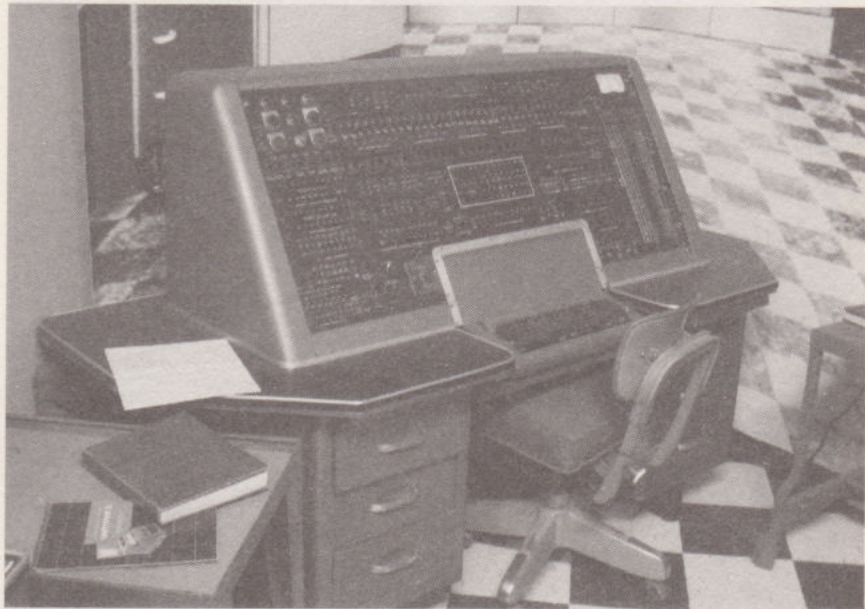


*The UNIVAC I control panel on display at the Museum of Science, Boston.*

For many of its users, the most significant progress represented by UNIVAC was not just its speed, but the fact that it used strips instead of punched cards for reading and storing information. Cards required human management, and replacing them with a strip represented an improvement in terms of automating the machine. Data was selected by the machine itself and UNIVAC could add two numbers in 0.5 microseconds.

Until the appearance of UNIVAC on the market, IBM had focused its efforts on selling calculators using punched cards. However, upon seeing the interest around the new computer, the company launched a new line for developing projects in that area. The first product of this line was the IBM 701, similar to UNIVAC, and which was referred to as an 'electronic data processing machine'. Von Neumann, who at that

point was building his computer at Princeton, provided advice on its creation. The IBM 701 was completed in 1952 and was sent to the Los Alamos atomic weapons laboratory at the start of 1953.

## The number π in the 20th century

The development of hardware throughout the 20th century resulted in new tools for calculating more precise approximations of the number π. The number has been calculated to more than one billion decimal places. The latest big approximation is correct to almost 2.7 billion decimal places, or $2.7 \cdot 10^{12}$.

However, prior to the invention of computers, the most accurate approximations had been calculated by the Englishman D. F. Ferguson, who by using calculators, managed to exceed one thousand decimal places. He calculated 620 decimal places in 1946, 808 decimal places in 1947, and 1,120 decimal places in 1949, working in partnership with John Wrench.

It was John Wrench who that same year calculated the first approximation of the number π using a computer. He used ENIAC to do so, upon a request from John von Neumann. The calculation took 70 hours and rendered 2,037 decimal places. Five years later, in 1954, Nicholson and Jeenel broke the previous record, calculating 3,092 decimal places in just 13 minutes using the IBM NORC, the most powerful computer at the time. In 1959, after another five-year interval, an IBM 704, the first mass-produced computer with hardware capable of floating-point arithmetic, calculated 16,167 decimal places in 4.3 hours. This time the calculations were carried out by François Genuys in Paris. It was not long before the 100,000 decimal place barrier was broken. This occurred in 1961, thanks to Daniel Shanks and John Wrench, using the new IBM 7090, which used transistors instead of vacuum tubes, making the machine six times faster than its predecessors. The computer took 8.7 hours to calculate 100,265 decimal places.

The spectacular figure of one million decimal places was broken by Jean Guilloud and Martin Bouyer in 1973, using a CDC 7600, manufactured by Control Data Corporation, one of IBM's competitors in producing second-generation computers (those using transistors with digital circuits). It took them 23 hours and 18 minutes to obtain precisely 1,001,250 decimal places. In 1966, Guilloud had set the record of 250,000 decimal places, obtained in 41 hours and 55 minutes, before setting a new record in 1967 with 500,000 decimal places calculated in 28 hours and 10 minutes. The 1980s belonged to Yasumasa Kanada and Kazunori Miyoshi, both from Japan.

In 1981 they managed to break the 2 million barrier in 137 hours; in 1982 they broke 8 million digits in 6 hours and 52 minutes; in 1983 the figure was 16 million in less than 30 hours; and in 1987, the Japanese computer, the NEC SX-2, allowed them to calculate over 100 million decimal places in a calculating time of 35 hours and 15 minutes. In 1989 Gregory Chudnovsky, regarded as one of the greatest living mathematicians, and his brother David, broke the one thousand million (billion) barrier on an IBM 3090.

The one billion barrier was broken again by Yasumasa Kanada and his team on a HITACHI SR8000/MPP. This occurred in Tokyo in December 2002. The figure of 1,241,100,000,000 decimal places was achieved after 600 hours of calculation (approximately 25 days), which represents a rate of 574,583 decimal places per second. In April 2009, Daisuke Takahashi, also from Japan, broke 2 billion decimal places at the University of Tsukuba, after just over 29 hours of computation. The current record of 2.7 billion decimal places belongs to the French computer programmer, Fabrice Bellard, who used a desktop PC running the Linux operating system, which took 131 days to carry out the calculation.

The majority of these results are based on the power series developed by the brilliant and enigmatic Indian mathematician, Srinivasa Ramanujan (1887–1920). One of them, published in 1914, made it possible to calculate eight new decimal places for each term in the series. The series is as follows:

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{[1103 + 26390n]}{396^{4n}}.$$

Ramanujan's results have been used to define series that converge quicker, obtaining various correct digits for each term. The series developed by the Scottish-born Canadian brothers Jonathan and Peter Borwein makes it possible to obtain 31 new digits for each term in the series.

The rest of the results, among which those of Yasumasa Kanada stand out, are based on a formula by Carl Friedrich Gauss (1777–1855) that establishes a relationship between the number $\pi$ and the arithmetic-geometric mean (AGM). Gauss' formula is as follows:

$$AGM\left(1, \sqrt{2}\right) = \frac{\pi}{\varpi},$$

where $\varpi = 2 \int_0^1 \frac{1}{\sqrt{1 - z^4}}\, dz.$

In this expansion, *AGM(a,b)* corresponds to the arithmetic–geometric mean of *a* and *b*.

The equations that have recently been developed by David Bailey, Peter Borwein and Simon Plouffe are the most interesting recent results on the number $\pi$. In 1997, the researchers published a series of formulae that made it possible to calculate the binary digit of $\pi$ at an arbitrary position, without the need to calculate the other digits. Clearly, this same method can be used to calculate digits in any base that is a multiple of two, particularly hexadecimal. The authors illustrated their method by calculating the hexadecimal digits of $\pi$ corresponding to the positions 1 million, 10 million, 100 million, 1,000 million and 10,000 million, which gave the hexadecimal digits shown below:

| Positions | Hexadecimal digits from corresponding position |
|---|---|
| 1,000,000 | 26C65E52CB4593 |
| 10,000,000 | 17AF5863EFED8D |
| 100,000,000 | ECB840E21926EC |
| 1,000,000,000 | 85895585A0428B |
| 10,000,000,000 | 921C73C6838FB2 |

## THE ARITHMETIC-GEOMETRIC MEAN

The arithmetic-geometric mean is formally defined by the convergence of two series, one formed by arithmetic means and the other by geometric means. Let's look at the expressions for both means:

$$AM(a,b) = \frac{a+b}{2},$$

$$GM(a,b) = \sqrt{ab}.$$

Hence, the first terms of the series *am* and *gm* are defined as: $am_1 = AM(a,b)$, $gm_1 = GM(a,b)$, which allow us to define the general terms as:

$$am_{n+1} = AM(am_n, gm_n),$$

$$gm_{n+1} = GM(am_n, gm_n).$$

Both series converge on the same value, the arithmetic-geometric mean: *AGM(a,b)*.

One of the formulae proposed by Bailey, Borwein and Plouffe is as follows:

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n}\left(\frac{4}{8n+1}-\frac{2}{8n+4}-\frac{1}{8n+5}-\frac{1}{8n+6}\right).$$

The term $16^n$ makes it possible for this expression to find binary digits. The following is another of their proposals:

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n}\left(\frac{-8}{8n+1}+\frac{8}{8n+2}+\frac{4}{8n+3}+\frac{8}{8n+4}+\frac{2}{8n+5}+\frac{2}{8n+6}-\frac{1}{8n+7}\right).$$

The calculation of the decimal places of $\pi$ has been at the centre of an impassioned pursuit that has occupied humankind's most distinguished minds for many centuries. Currently, thanks to the help of computers, the number of decimal places known for $\pi$ is in the billions. However, most calculations only require a few decimal places.

## NUMBERING SYSTEMS

As indicated by its name, the decimal numbering system uses ten different digits. The most common notation is 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

Binary notation uses just two: 0 and 1. Hexadecimal notation uses 16. The most commonly used symbols are: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. The value of the symbol A is 10 in decimal notation; the value of B is 11, C is 12, D is 13, E is 14 and F is 15.

Binary and hexadecimal notation are closely related: 16 is a multiple of 2 and it is extremely easy to switch between one and the other.

To convert a number from binary to hexadecimal notation, its bits should be grouped into sets of 4. There are 16 possible permutations of 0s and 1s in sets of 4. Each set of four binary digits corresponds to a single hexadecimal one.

To convert a number from hexadecimal to binary notation, each hexadecimal digit will correspond to 4 binary digits, based on the following equivalences:

| | | | |
|---|---|---|---|
| 0000: 0, | 0001: 1, | 0010: 2, | 0011: 3 |
| 0100: 4, | 0101: 5, | 0110: 6, | 0111: 7 |
| 1000: 8, | 1001: 9, | 1010: 10, | 1011: 11 |
| 1100: 12, | 1101: 13, | 1110: 14, | 1111: 15 |

In an article published in 1984 in a specialist journal, the Borwein brothers did not justify their pursuit, nor ask why; they only stated their own amazement:

"It requires a mere 39 digits of $\pi$ in order to compute the circumference of a circle of radius $2 \times 10^{25}$ (an upper bound on the distance travelled by a particle moving at the speed of light for 20 billion years, and as such an upper bound for the radius of the universe) with an error of less than $10^{-12}$ meters (a lower bound for the radius of a hydrogen atom). There is no doubt that calculating the number $\pi$ to the greatest possible precision has a mathematical significance that extends beyond its usefulness."

# Chapter 5

# Programming and Software

The development of computing hardware has run in parallel to the evolution of programming languages. Essentially, a programming language can be defined as a language that allows us to tell a computer how to behave to find the solution to a given problem. It forms a list of ordered steps that must be carried out to provide the requested result, expressed in a way that can be understood by the computer. This definition immediately brings to mind the well-known concept in the history of mathematics that we saw in the first chapters: algorithms, the ideal tools for such a job. Effectively, the more formal definition of a programming language is the language that describes to a computer the algorithms directing its operation.

It goes without saying that the descriptions proposed by a programming language must be rigorous, without grounds for ambiguity, and must always be designed to solve a specific problem. The notation must allow for a fundamental feature of algorithms and programming languages: repetition. Programming languages have two methods available for implementing repetition: iteration and recursion. Iteration is explicit repetition expressed using instructions such as *repeat*, *while* and *for*. Recursion is implicit repetition, in which part of an action includes repeating itself, implemented by having procedures refer to themselves.

Throughout this book, it has been shown that the term algorithm arose long before computers. From its origins in pure mathematics, the word only referred to the description of procedures for arithmetical calculations, and it was only later that

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### THE WORD 'IMPLEMENT'

Implement means to put into operation, to apply suitable methods for carrying out a task.

In computing, the term means to program a given algorithm in a specific language.

## THE WORD *PROGRAM*

The use of the term 'to program' to define the action of specifying the actions to be carried out by a computer has its roots in the group that created ENIAC at the Moore School of Electrical Engineering, at Pennsylvania University. Until then, the most commonly used term was to set up, since ENIAC (in the photograph below) was programmed by changing connections and flicking switches, i.e. by directly modifying the wiring of the machine. However, as the separation between software and hardware gradually began to emerge, the term was replaced by the idea of "programming".
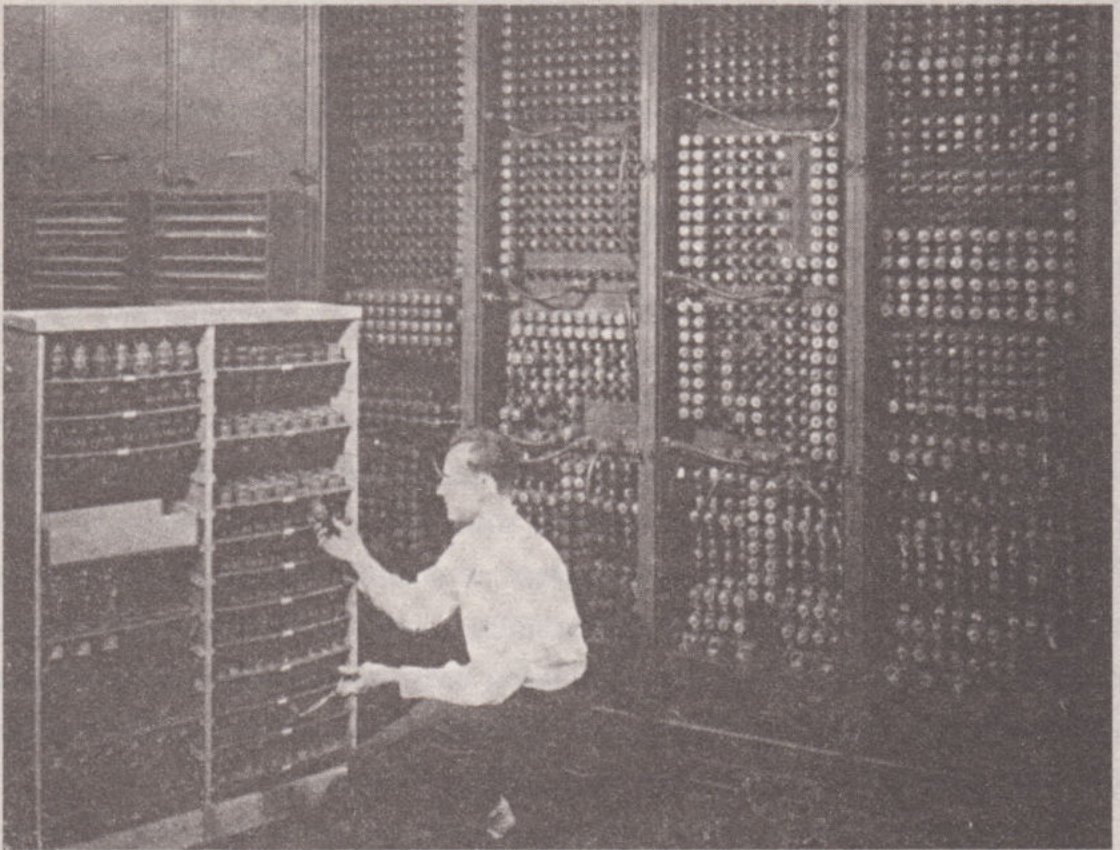


it adopted this more generic meaning, now closely related to computing, that is so popular today. However, programming languages are merely the evolution of those descriptions, in pursuit of the highly precise formalism that is essential in allowing computers to understand.

The oldest algorithms, which hail from Babylon, were for solving algebraic equations and were written in a general format into which actual data could be inserted. They lacked both iteration and conditional expressions of the form

'if x < 0 then', since the Babylonians were not aware of the number zero. To express more than one possibility, Babylonian mathematicians repeated the algorithm as many times as required. Many centuries had to pass before Euclid would

## IMPLEMENTING EUCLID'S ALGORITHM

By means of example, the implementation of the greatest common divisor of two numbers *A* and *B*, is provided below, first in PROLOG and then in Java. The abbreviation *gcd* stands for 'greatest common divisor' (or greatest common denominator).

The PROLOG implementation uses three rules, corresponding to the three possible cases. In all cases, the first two arguments are numbers and the third argument can be interpreted as the result. The first rule corresponds to the case when the second argument is zero; the second rule applies when the first argument is greater than the second, and the third when the second is greater than the first.

```
gcd(A, 0, A).
gcd(A, B, D) :- (A > B),(B > 0),R is A mod B,gcd(B,R,D).
gcd(A, B, D) :- (A < B),(A > 0),R is B mod A,gcd(A,R,D).
```

The Java implementation is also based on the rules above. It takes two numbers (A and B) as its input parameters and returns the greatest common divisor as the result. The first version is recursive and the second iterative.

```
public static int gcd (int A, int B) {
        if (B == 0) { return A; }
        else if (A > B) { return gcd(B, A % B); }
        else if (A < B) { return gcd(A, B % A); }
        return 1;
}
public static int gcdIterative (int A, int B) {
        int r = 0;
        while (B > 0) {
            r = A % B;
            A = B;
            B = r;
        }
        return A;
}
```

discover an algorithm for calculating the greatest common divisor of two numbers around 300 BC. Euclid's algorithm, as it is still known, is generally implemented using recursion.

However, these algorithms do not form part of a continuous evolutionary process but represent isolated cases. As for procedures for carrying out automated tasks, the great historical landmark was the programming of looms, which began with Jacquard. Jacquard's loom used punched cards to determine the pattern of a textile. Hence, in a primitive way, the cards contained what we could call a 'programme' for the machine's execution.

Charles Babbage applied the idea of punched cards to computing in his early machines. From a modern perspective, it is understood that those primitive programmes were already written in machine language, making Ada Byron the first programmer. However, the notion of a programme stored in memory had yet to be invented.

In spite of these early attempts, and the theoretical work of the 1930s and 1940s on the lambda calculus and the Turing machine, the description of algorithms was not fully developed until the arrival of the first computers: Colossus, Mark I, ENIAC, EDSAC and UNIVAC. Programs, or *software*, stored in memory arose as a solution for simplifying and reducing the time spent on directly re-configuring the *hardware*, the method of programming used to that point in time.

The first computers were programmed in octal (base 8). Some of the first languages that allowed symbols to be represented were John Mauchly's Short Order Code (1949) and Betty Holberton's Sort-Merge Generator. The former originally ran on a BINAC as an interpreted language. The routines corresponding to the symbols were stored in memory and invoked by the system. UNIVAC inherited this system. Programs written in this interpreted language were 50 times slower than their counterparts using machine code.

For its part, the Sort-Merge Generator was an application developed for UNIVAC, which, given the files on which it had to operate, created a program in machine code for ordering and merging files that contained the input and output operations.
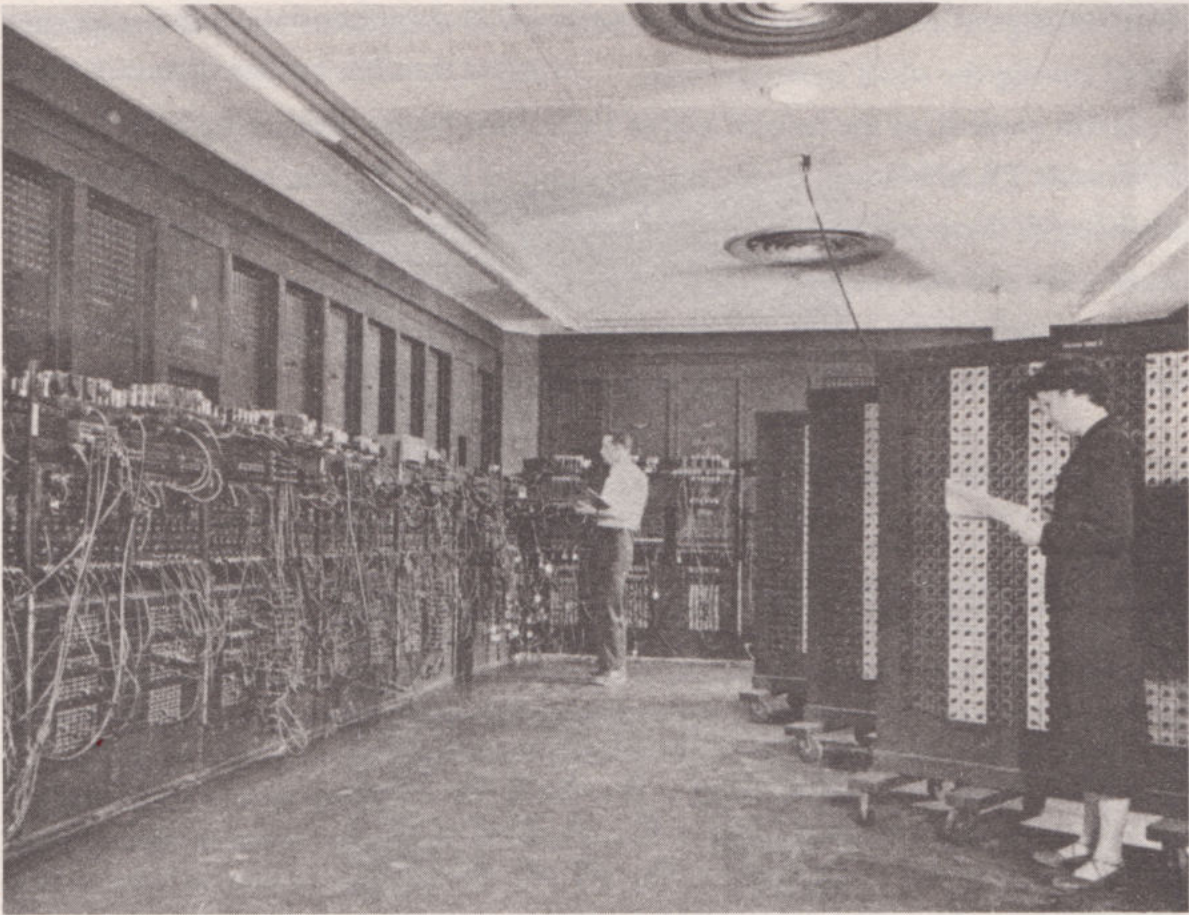
*Betty Holberton, who appears in this photograph carrying out control work for ENIAC, developed one of the first programming languages.*

These first automatic programming systems, were limited to providing easily memorable operation codes and symbolic addresses, or retrieving subroutines from a library and inserting the code after replacing the addresses with the operands. Some systems permitted the interpretation of floating point operations and indexing. However, with the exception of the A–2 compiler and Laning and Zierler's algebraic system, until 1954 not even the most powerful systems provided more than a synthetic machine with code that differed from that of the real machine.

However, the problems related to that model of operation were not just technical but financial. The cost of the programmers running a computing centre increasingly exceeded the cost of the computer itself, since the price of the technology was falling and, with it, the prices of computers. Furthermore, between 25% and 50% of the time a computer was 'in use' was spent on programming and debugging. Automatic programming systems cut the speed of the machine by a factor of 5 or 10. The companies selling the systems exaggerated their performance in the oldest tradition

of marketing to such an extent that a widespread scepticism towards computing spread among the small customer base.

In this context, in the second half of 1954, IBM began developing FORTRAN (FORmula TRANslation) under the direction of John Backus, with the intention of solving all these problems. The compiler focused above all on creating an efficient target code, and was successful in this respect. In fact, the quality of the target code and the transformations carried out to obtain an efficient program surprised even those who had taken part in its implementation.



*A punch card prepared using the FORTRAN language.*

With the advent of FORTRAN, the programmer was able to use a precise language to specify mathematical procedures. As the language permitted a certain level of abstraction, the code could be translated for use on different machines. This gave rise to the first abstractions of data: information was stored anywhere in the memory and no longer appeared stored as a sequence of bits, but instead as an integer or a real number. The IBM language saw the appearance of the basic constructs of imperative languages: the conditional (IF <condition> jump-true jump-false) and the loop (DO end-loop variable=start, end, step).

FORTRAN was followed by a series of languages that also used the idea of data abstraction: ALGOL-60 (one of whose creators was the Dutch scientist Edsger Dijkstra), COBOL and LISP (forerunner of functional languages, which we shall discuss in more detail further on). Each included the data types required for the applications to be built. These languages were specifically designed for solving certain problems. However, PL/I, on the other hand, was designed as a general-purpose

language, and as such included all the innovations from previous languages, which made it enormous and difficult to learn.
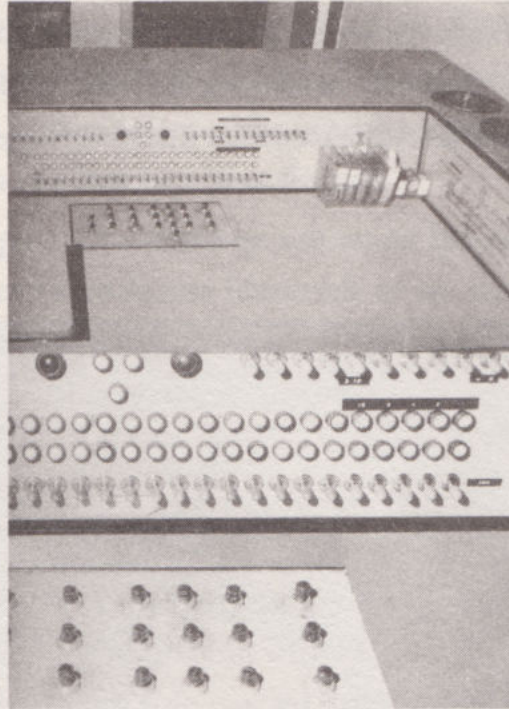


*The Electrologica X1, which operated between 1958 and 1965, used the Algol-60 language.*

The next generation of languages was less ambitious and more effective. It included Simula-67 and Pascal. Instead of having an exhaustive set of *a priori* abstractions, the languages had flexible mechanisms suitable for defining new abstractions. Pascal and ALGOL-68 made it possible to define new types, using the basic predefined constructors (*array*, *record*, etc.). These new types could be viewed as an abstraction built on top of an internal representation and with an associated set of operations. However, in spite of its flexibility, the model had a problem. While it was not possible to access the representation of the predefined types, i.e. a predefined object could not be manipulated directly (it was only possible to do so through operations), it was, however, possible to access the structure of defined types and modify their values. This was due to the fact that the language did not distinguish between the two levels of abstraction – the level at which the program used a type as an object and the level at which it was implemented. This confusion made it difficult to create programs, correct errors and make any modifications. When the programs reached a certain size, the task became insurmountable.

The solution came to hand with abstract data types, and the languages that supported them (Ada, Modula-2 and CLU), distinguishing between the two levels and hiding information. At the level at which the program used the type as an object, it was not possible to access the structure that implemented it. This could only be manipulated through the set of specific operations defined in the interface. The representation was completely hidden. At the level of the implementation, the specific operations were defined as part of the interface, together with the structure that supported them.

As the programmer would know the operations for a type and their behaviour, but not their representation, he or she could reason in terms of an abstraction. Any change to the implementation that did not modify the interface would not affect the modules that used it, since these were unable to access the representation, only the operations of the interface.

These mechanisms for abstraction allowed the languages to structure programs in terms of objects (represent the type together with its operations). This type of structuring was optional in some languages, such as Ada and Modula-2, although it became compulsory in others. In CLU, the programmer had to group the application data in classes, referred to as *clusters*. This was also the case with object-orientated languages, which implemented, together with objects, the concept of inheritance that made it possible to define an object based on another predefined object.

Object-oriented programming is a programming paradigm that uses objects and their interactions to design computer applications and programs. It was designed with two aims: to implement programs on a large scale and model human reasoning in artificial intelligence. In terms of the latter, they have contributed to the development of techniques for structuring knowledge by grouping the information and properties relative to a given concept in a single entity.

The first language that grouped data and procedures into a single entity was Simula-I. As its name suggests, it was designed for simulations by the Norwegian Computing Center (NCC) under the direction of the Norwegian mathematician and politician Kristen Nygaard. The first version was completed in 1965. Its successor was a general-purpose language called Simula-67, which formalised the concepts of object and class, and also introduced inheritance. Years later, Smalltalk-80, which descended from Simula through two previous versions (Smalltalk-72 and Smalltalk-76), generalised the concept of an object and transformed it into the only entity manipulated in the language. Smalltalk can be regarded as a pure object-oriented language. It regarded classes of objects as objects too, and the control structures were operations on the

appropriate classes. At the start of the 1970s, the Xerox Palo Alto Research Center, known as Xerox PARC, began to develop the Dynabook system, a personal tool for information management, with a windows-based interface, text menus and icons, or rather a graphical user interface (GUI) along the same lines of the ones we use today. The Dynabook was the brainchild of the American Alan Kay and was originally devised for introducing children to the world of computers. It was written in BASIC and completed in 1972. It implemented the message-passing mechanism, in addition to the concepts of class and instance from Simula.



*Alan Kay being awarded the title of doctor Honoris Causa by the University of Murcia in Spain in recognition of his contribution to the development of computing. The ceremony was held on 28 January, 2010.*

At present, there are many object-oriented languages (Eiffel, C++, etc.), some of which are extensions of other languages. C++ is an extension of C developed by the Dane Bjarne Stroustrup, implementing classes from Simula. CLOS was developed to standardise the object system for Common LISP. The ideas of objects and inheritance have been used in the field of artificial intelligence to develop languages based on *frame* languages such as *KRL* and *KL-ONE*, and actor languages such as *Act1*, *Act2*, *Act3* and *ABCL/1*.

Abstraction and objects have appeared in all recent languages, both in object-oriented languages in which they are defined directly, such as Java and Python, and in procedural languages that implement object-oriented constructs, such as PHP. However, languages have also been developed for the purpose of building applications quickly, as well as scripting languages. This is the case with PHP and JavaScript, both developed in the final decade of the 20th century. These languages aim to make it quicker and easier to create programs. This is certainly the case when it comes to small programs, but in comparison with the previous languages, the design of large-scale programs is more complex. At any rate, the influence of object-oriented languages on program design has resulted in new tools, such as so-called modelling languages like UML.

## The functional paradigm

In imperative languages, computation is implemented by modifying variables using assignments. A program written in an imperative language is limited to the structure of the von Neumann machine, in the sense that there are cells with values. An assignment to a variable is essentially nothing more than changing the value of a cell. In functional languages, results are obtained by applying functions defined by composition or recursion.

The roots of functional languages lie in work by John McCarthy, who was responsible for the term 'artificial intelligence' and worked at the Massachusetts Institute of Technology, published in 1960 in *Communications of the ACM*, the monthly journal of the veteran Association for Computing Machinery (ACM), the society responsible for awarding the Turing Prize.

It all began in 1958, when McCarthy studied the use of operations on linked lists for a program for symbolic differentiation. As differentiation is a recursive process, it made use of recursive functions. Furthermore, he believed it would be useful to pass functions as arguments to other functions. The project to implement the new language began in autumn that same year. The result was published two years later in an article entitled *Recursive Functions of Symbolic Expressions and Their Computation by Machine, Part I* (part II was never published), and represented the first version of LISP (*List Processing*), the first functional language that pioneered many ideas in the field of computer science. McCarthy used Alonzo Church's Lambda calculus notation to define it.
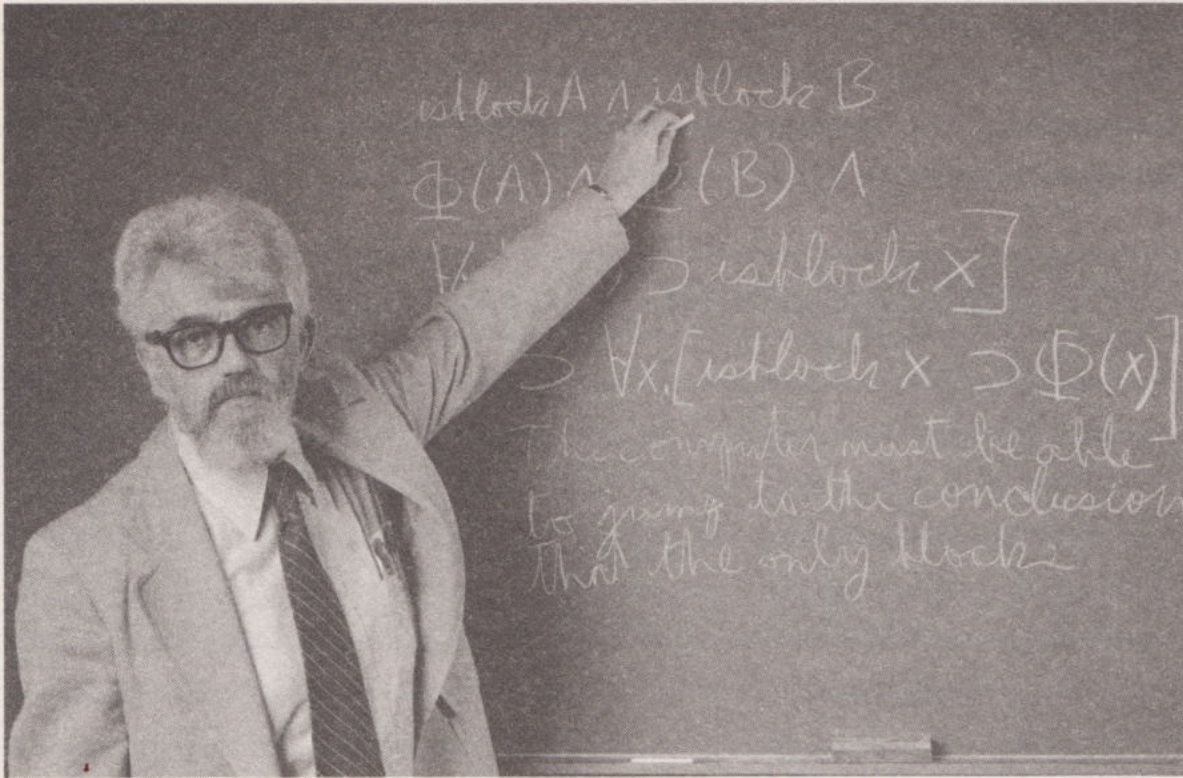
*John McCarthy, inventor of the term 'artificial intelligence', in a photograph taken in 1980s at the University of Stanford in California.*

## THE LAMBDA CALCULUS

The lambda calculus was defined by Church and Kleene at the start of the 1930s. It expresses the same things as a Turing machine, despite operating differently. From a formal point of view, the lambda calculus used expressions and expression-rewriting rules that modelled the application of functions or computation. Consider the definition of true and false as an example:

true: λxy.x

false: λxy.y

Hence, the 'AND' function is defined as:

AND: λpq.p q p

As such, to calculate 'AND true false,' we must replace each term by its equivalent expression in the lambda calculus:

(λpq.p q p) (λxy.x) (λxy.y)

We must then apply the rewriting rules, which give us the expression (λxy.y), equivalent, as we have noted above, to false. Numbers and numerical operations are defined in a similar manner.

LISP represented data and programs in the same way. It made use of recursion as a fundamental control structure and avoided the lateral effects of imperative programming. The language also implemented conditional expressions and Lukasiewicz' prefix (or Polish) notation.

## PREFIX OR POLISH NOTATION

Mathematical expressions are expressed with the corresponding symbol for the operation placed before the operands. Hence, we have

$$(a + b) - (c \cdot d)$$

which translates into Polish notation as:

$$- + ab \cdot cd.$$

In the second half of the 1960s, Peter Landin designed a new functional language, ISWIM (If You See What I Mean), based on LISP and the lambda calculus. ISWIM spawned an entire family of functional languages, including ML, FP and Miranda. At that time functional programming was still only of interest to a small number of researchers, however it began to receive greater attention from 1978 onwards, when John Backus, the creator of FORTRAN, published his article *Can Programming Be Liberated from the von Neumann Style?* Backus fiercely criticised conventional programming languages and advocated the development of a new paradigm, which he called functional programming and which placed the emphasis on functionals (functions that operated on other functions). In his article, which earned him the Turing Prize, he described the FP (Functional Programming) language, in which there were no variables. That wake-up call aroused the interest of researchers in functional languages, and precipitated the appearance in new languages.

At present, there are two large families of functional languages: LISP-style languages and ISWIM-style languages. The former includes dialects of LISP, such as Common LISP, as well as languages in their own right, such as Scheme. The second family includes Standard ML, the fruit of a process of standardisation based on ML and Hope, both developed at the University of Edinburgh. ML is a strongly-typed functional language, in contrast to LISP, which means that any expression in the language has a type that can be deduced by the system at compilation time (a static type). Moreover the programmer can incorporate new types into the system using

mechanisms for defining abstract data types. ML permits the definition of modules and generic modules (functors). In contrast to ML, in Hope definitions require an explicit type declaration.

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## FUNCTIONAL LANGUAGES: EXAMPLES OF IMPLEMENTATION

The following examples correspond to the definition of the factorial function and show the syntactical similarities of the two large families of functional languages. In LISP languages (Scheme, Hope and ML) there are variables, albeit functional ones, and the definition of the factorial function is recursive, similar to the example in Java shown above. In the FP language, however, there are no variables. The FP definition uses the *iota* function, which, when applied to a number, returns the list of all natural numbers from one to that number. The construction /* is applied to this list, which multiplies its elements. Formally, /op extends a binary operation to a list.

Definition in LISP:

```
(defun factorial (n)  (if (= n 0) 1 (* n (factorial (− n 1)))))
```

Definition in Scheme:

```
(define factorial
    (lambda (n)
        (if (= n 0) 1 (* n (factorial (− n 1))))))
```

Definition in Hope:

```
dec fact : num − > num;
− − − fact 0 < = 1;
− − − fact n < = n*fact(n−1);
```

Definition in ML:

```
fun f (0 : int) : int = 1
    If (n : int) : int = n * f (n−1)
```

Definition in FP:

```
fact ≡ /* op iota
```

## INFINITE LISTS IN HASKELL

The following definitions of two infinite lists in Haskell help us to understand the difference between eager and lazy evaluation. The definitions are recursive, which means they contain calls, or references, to themselves.

The first corresponds to the list of natural numbers. By induction, it is assumed that the list is already correctly defined. Hence, it increments all the elements of the list by one, which gives the list 2, 3, 4..., to which the number 1 is added. In the definition, all members of the list of natural numbers are incremented using the 'map (+1) naturals' construct.

The second defines the Fibonacci numbers. Assuming the list already exists, the construction of the numbers is based on aligning each number with the following and adding the two aligned numbers In the definition, 'listfibs' is aligned with the tail of listfibs, calculated using 'tail listfibs'. Then the pairs that have been formed are added to a number from each list, using 'zipWith (+)'.

$$naturals = 1 : map (+1) naturals$$
$$listfibs = 0 : 1 : zipWith (+) listfibs (tail listfibs)$$

These definitions work because of lazy evaluation. The eager evaluation used by the majority of languages would leave the computer thinking for an infinite period of time in order to evaluate the expressions. Note that the definition of the natural numbers first requires the definition of the infinite list, before adding one to all its elements.

The languages LISP and ML implement 'eager evaluation', which means that all the arguments of a function are evaluated prior to its application.

However, there are also languages without evaluation, or with 'lazy evaluation', such as Haskell, Lazy ML, some versions of Hope, and above all Miranda, defined by David Turner based on KRC and SASL. In these languages, an argument is only evaluated when it is necessary to determine its value, making it possible to write programmes that would never terminate when using eager evaluation.

For example, it is possible to define an infinite list of numbers that will never be fully evaluated, instead only evaluating the terms required at a given moment.

λ **Miranda™**

*A Non-strict Polymorphic Functional Language*

Miranda is a pure, non-strict, polymorphic, higher order functional programming language designed by David Turner in 1983-6. The language was widely taken up, both for research and for teaching, and had a strong influence on the subsequent development of the field, influencing in particular the design of Haskell, to which it has many similarities. Miranda is however a simpler language. Here is a short description. For more detail you can read the Overview paper below, look at these examples of Miranda scripts, or read the definitions in the Miranda standard environment.

DOWNLOADS *includes Linux, Windows(Cygwin), Intel/Solaris, SUN/Solaris & Mac versions.* To be informed of new versions add yourself to the mailing list.

Why the name Miranda?

Browsable version of Miranda system manual

Browsable version of UNIX manual page mira.1

### Papers about Miranda

The following paper covers the main features of the language in around a dozen pages. First published in 1986, it remains the best introduction to Miranda:

D. A. Turner *An Overview of Miranda*, SIGPLAN Notices 21(12):158-166, December 1986. **PDF** [139K]

This earlier description of Miranda contains a more detailed discussion of algebraic and abstract data types, including algebraic data types with laws:

D. A. Turner *Miranda: A Non-Strict Functional Language with Polymorphic Types*, Proceedings IFIP Conference on Functional Programming Languages and Computer Architecture, Nancy, France, September 1985 (Springer Lecture Notes in Computer Science 201:1-16).

The first published account of functional programming in Miranda is the following (no electronic version exists):

D. A. Turner *Functional Programs as Executable Specifications* in proceedings of a meeting of the Royal Society of London on 15 February 1984, published as *Mathematical Logic and Programming Languages* pp 29-54, eds Hoare and Shepherdson (Prentice Hall, 1985).

*The homepage of the Miranda website, the programming language developed by David Turner.*

## The logic paradigm

When the families of imperative and functional languages had already been established, another alternative sprung up. The third paradigm is based on logic languages. Logic programming consists in the application of philosophical logic to programming language design, the formal science that studies the principles of proof and valid inference. It should not be confused with computational logic, which is mathematical logic as applied to computer science.

The imperative and functional paradigms conceived of a program as a function that calculates a result based on given input values. In imperative programming, for example, the program is a command that reads certain input files and, after having carried out the calculations, writes the output to other files. In a logic language, on the other hand, the program implements a relationship. Using a set of rules referred to as Horn clauses, the programmer establishes the facts that are true, allowing the

user to ask if a relationship is true or false. The computation is based on Robinson's resolution principle, and consists of evaluating if the relationship given by the user is true or false, or establishing under what circumstances it holds.

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## HORN CLAUSES

Horn clauses are a set of logic rules of the type "if a condition is true, then a consequence is also true". They can be viewed as implications with a conjunction of premises and a single consequence. There can be zero, one or more premises:

$$a1 \wedge a2 \wedge ... \wedge aN \rightarrow b.$$

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The logic language par excellence is PROLOG, whose name is derived from *PROgrammation en LOGique*. It was developed in 1972 and is the only language in this family still in use today. The first version was designed under the supervision of the researcher Alain Colmerauer at the Artificial Intelligence group at the University of Aix-Marseille, in partnership with the British logician Robert Kowalski, from the University of Edinburgh. The language arose as the result of the convergence of two lines of research. The first, Colmerauer's, was not directly related to computing, but instead studied the processing of natural language. The second, Kowalski's, studied automatic proofs of theorems. The language was influenced by W grammars, a

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## A DEFINITION IN PROLOG

As an example of programming in PROLOG, let us consider a small program for calculating the natural numbers. To do so, we define nat(*N*) such that it is only true when *N* is a natural number. The definition is constructive in the sense that the language will calculate natural numbers when we ask what values of *N* satisfy the predicate. The program is as follows:

nat(*N*): −*N* = 0.
nat(*N*): −nat(*Np*), *N* is *Np* + 1

The first line states that zero is a natural number. The second line states that if *Np* is a natural number, *Np* + 1 must also be natural. More formally, the program decides if *N* is a natural number when *N* is zero or there is a natural number *Np* such that *N* is *Np* + 1.

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notation used to describe ALGOL-68, and the Planner language, developed at the University of Stanford. The success of PROLOG was due to the implementation developed by David Warren, who formed part of Kowalski's group at Edinburgh. The so-called Warren Abstract Machine (WAM) executed programs at a speed comparable to that which could be reached by programs written in LISP.

## The formal description of programming languages

The syntactical and semantic description of the first programming languages was achieved using informal methods. The scientific community began by considering syntax. In 1960, in order to describe the syntax of ALGOL-60, John Backus and Peter Naur devised the notation known as BNF (Backus–Naur Form), which, in addition to being useful for formally describing the syntax of languages, made a considerable contribution to their design. A few years after its development, significant similarities were discovered with the grammatical rules established in the 4th century BC by Pānini, which described classical Sanskrit.

At the same time as BNF was being defined, the famous linguist, philosopher and political analyst Noam Chomsky developed his theory of grammars, known as the Chomsky hierarchy. His theory classified grammars and the languages they generated into four types, depending on their expressive power. Type 3 refers to regular grammars, the most restrictive. Type 2 refers to context-free grammars, which can be described using BNF. Type 1 refers to context-sensitive grammars, and type 0 to unrestricted grammars, both having a greater power of expression than the preceding categories. The Chomsky hierarchy is related to computing power. A general computing system will be able to recognise sentences from any language expressed using a type 0 language. The Turing machine and the lambda calculus belong to this category of computing systems.

Years later, the Swiss recipient of the Turing Prize, Niklaus Wirth, who designed numerous languages, specified the syntax of the Pascal language, introducing syntax diagrams and an extension of BNF, EBNF (Extended BNF). Although these notations did not increase the expressive power of BNF, they are currently widely used, since there are programmes that automatically generate syntax recognition engines from descriptions of syntax.
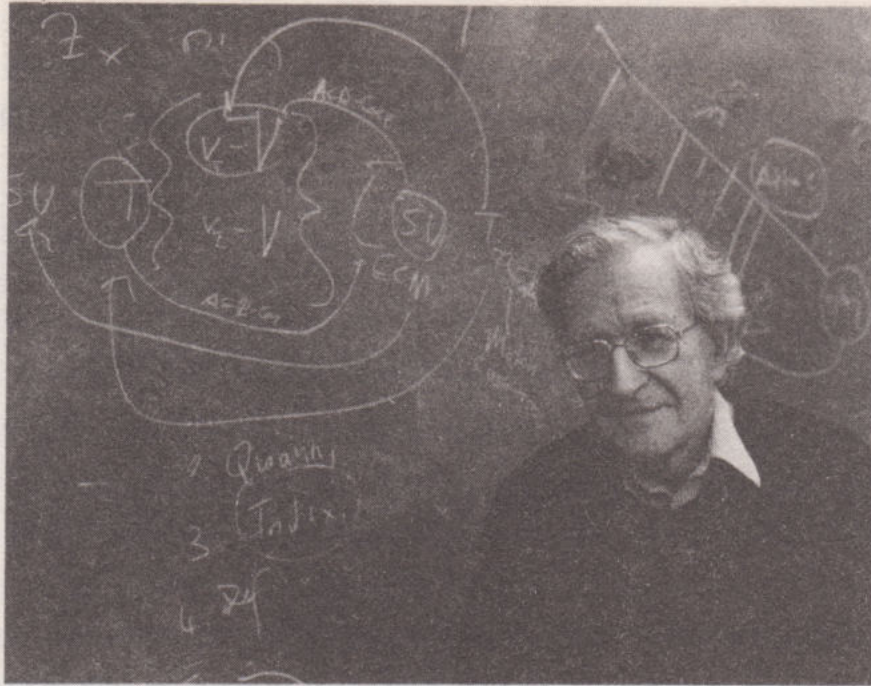
*The linguist Noam Chomsky, who devised the hierarchy that bears his name.*

The formal specification of the semantics of programming languages, or rather how languages should behave, has been less successful than syntax. So far, a wide range of formal approaches have been developed, but none has achieved the popularity of the models developed for syntax.

The first proposal was VDL (Vienna Definition Language), developed at IBM's Vienna Laboratory to provide a formal specification of the PL/I language. It consists of two parts: The first is a translator that builds an abstract syntax tree for each PL/I program; the second is an interpreter that specifies how the program represented by the tree should be executed or interpreted. This type of semantics is called operational and intrinsically requires a high level of detail. As the PL/I language is extremely large, irregular and full of individual cases, its formal specification resulted in a colossal document that was difficult to understand, which earned it the ironic nickname the VTD (short for the Vienna Telephone Directory). However, in spite of this, it represented an important achievement in this field.

The Vienna Laboratory continues with the research to develop a second, improved solution: the VDM (Vienna Development Method), which included a number of *ad hoc* properties for specifying imperative programming languages. It was developed in 1982 as a result of the combination of the two visions of Dines Bjørner and Cliff Jones that would give rise to the two schools of thought in programming, the Danish

school and the English school. VDM was used for specifying Pascal and ALGOL-60, as well as for a subset of Ada'79.

On the other hand, the distinguished American computer scientist, Robert Floyd showed in 1967 how it was possible to reason about the correctness of a program by associating assertions with the arcs in its flow diagram. Each assertion is a logical formula that expresses a relationship between the value of the program variables in order to establish an assertion that is true until the end of the program and expresses a relationship between its inputs and outputs. Floyd's techniques were refined and improved by the British logician Charles Hoare, who expressed them as a system of axioms and rules of inference associated with the constructs of programming languages, hence defining axiomatic semantics. In 1973, Hoare and Wirth published the axiomatic specification for a subset of Pascal. During its construction, they found certain irregularities in the original design of the language, which led them to an improved design. The following year, Hoare and Lauer explored the complimentary use of axiomatic and operational semantics. Edsger Dijkstra introduced the use of the weakest precondition in 1975.



*Donald Knuth (left) and Herman Zaph discussing the properties of a new computer font in this photograph taken at California's Stanford University in 1980.*

There are other ways of describing the semantics of a language. Attribute grammars appeared in 1968, developed by one of the most famous experts in computer science, well-known for his sense of humour, Donald Knuth. The grammars were extensively studied in relation to compilation techniques. However, there is also a fourth type of semantics, denotational semantics, developed at the University of Oxford by the American, Dana Scott, and the Briton, Christopher Strachey, at the start of the 1970s. Denotational semantics assigns a meaning to each programme (a denotation) in terms of a mathematical entity, which is normally an application of input values to output values. Systems have been studied for generating compilers based on the denotational semantics of a language. However, thus far they have proved extremely inefficient.

Hence, the history of computing is far from having reached the culmination of all the preceding effort and knowledge. On the contrary, its evolution continues, bound more tightly than ever to technological developments that offer undreamed of perspectives. Computer programming languages no longer just configure our computers, but also our televisions, mobile phones, and even the most trivial of household devices. Once more, we are taking our first steps in a new world. However, as we have seen, we have not arrived here as the result of sheer coincidence, but of the effort made by humankind in developing knowledge, an endeavour that is for now merely an episode in a journey towards an unimaginable, and no doubt amazing, future.

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# Index

# From the Abacus to the Digital Revolution
## Counting and computation

Throughout their evolution, calculating tools have always been the result of the available technology and the numbering system in popular use at any given time. From prehistoric accounts of the Roman abacus and Arabic algorithms, to the first automatic calculators, the history of calculation is also largely a history of numbering systems. The most recent milestone in this ongoing evolutionary process has been the development of modern-day computers and computing systems, which are continually being refined towards the same goal: to provide increasingly powerful tools to carry out ever more complex calculations.